Simplicial Computations in Topology

by
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Outline

1. Introduction
2. Adaptation points of MMarco's method
3. The effective method for $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$
4. Examples
5. Conclusions

Manero & Marco [2020]

Effective computation of degree bounded minimal models of CDGA's

Given a finitely presented Commutative Differential Graded Algebra (CDGA):

Manero & Marco [2020]

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Given a finitely presented Commutative Differential Graded Algebra (CDGA):

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(3) A specific implementation of the method is given.

(4) Also, two criteria for i-formality, one necessary and one sufficient, are provided.

Manero & Marco [2020]

Open questions

(1) Can we adapt this method to compute effectively the minimal model of a topological space?

(2) Is the equality of numerical invariants up to degree i a sufficient condition for i-formality?

(3) Is the Ψ-condition up to degree i necessary for i-formality?

The aim of this thesis

To answer the question

(1) Can we adapt this method to compute effectively the minimal model of a topological space?

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The aim of this thesis

This work answers the question in a positive way...for some kind of spaces.

The proof is the method itself

We give a constructive and effective method to compute the Sullivan minimal model **of a finite type simplicial set**. Also, an implementation in a Computer Algebra System of such method is given.

Commutative Differential Graded Algebras

Definition of a CDGA

A commutative differential graded algebra (or CDGA), $\mathcal{A}$, over $\mathbb{Q}$ is a graded $\mathbb{Q}$-algebra, $\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}^i$, together with a $\mathbb{Q}$-linear map $d : \mathcal{A} \to \mathcal{A}$ that satisfies the following conditions:

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(3) $d \circ d = 0$

(4) $d(x) \in \mathcal{A}^{i+1},\ \forall x \in \mathcal{A}^i$

(5) $d(xy) = d(x)y + (-1)^i x d(y),\ \forall x \in \mathcal{A}^i,\ \forall y \in \mathcal{A}$

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(3) The differential of a generator a_s is expressed in terms of the generators a_l's with $l < s$.

Morphisms of CDGAs

A morphism of graded commutative algebras,

$$\phi : \mathcal{A} \to \mathcal{B},$$

is said to be a *morphism of commutative differential graded algebras* if it commutes with the differential. In particular:

$$\phi(d_{\mathcal{A}}(xy)) = d_{\mathcal{B}}(\phi(x)\phi(y)),\ \forall x, y \in \mathcal{A}.$$

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Definition

When the morphism ϕ additionally satisfies the following conditions:

(1) $\phi^* : \mathcal{H}^j(\mathcal{A}) \to \mathcal{H}^j(\mathcal{B})$ is an isomorphism for every $j \leq i$

(2) $\phi^* : \mathcal{H}^{i+1}(\mathcal{A}) \to \mathcal{H}^{i+1}(\mathcal{B})$ is injective.

then, it is said that ϕ is an i-**quasi-isomorphism.**

Minimal models

Definition

An i-minimal model of a CDGA, $(\mathcal{A}, d_{\mathcal{A}})$, is an i-minimal CDGA, $(\mathcal{M}, d_{\mathcal{M}})$, together with an i-quasi-isomorphism,

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A minimal model of a CDGA, $(\mathcal{A}, d_{\mathcal{A}})$, is a differential graded algebra $(\mathcal{M}, d_{\mathcal{M}})$ and a quasi-isomorphism ϕ that satisfy:

(1) $(\mathcal{M}, d_{\mathcal{M}})$ is i-minimal $\forall i \geq 0$,

(2) ϕ is an i-quasi-isomorphism $\forall i \geq 0$.

Existence and uniqueness of Sullivan minimal models

Definitions [Griffiths & Morgan]

A CDGA $(\mathcal{A}, d_{\mathcal{A}})$ is said to be *connected* if $\mathcal{H}^0(\mathcal{A}) = \mathbb{Q}$.
Additionally, if $\mathcal{H}^1(\mathcal{A}) = 0$, it is said that $\mathcal{A}$ is *simply-connected*.

Theorem [Sullivan, Morgan]

Every simply-connected commutative differential graded algebra $\mathcal{A}$ has a minimal model, unique up to isomorphism.

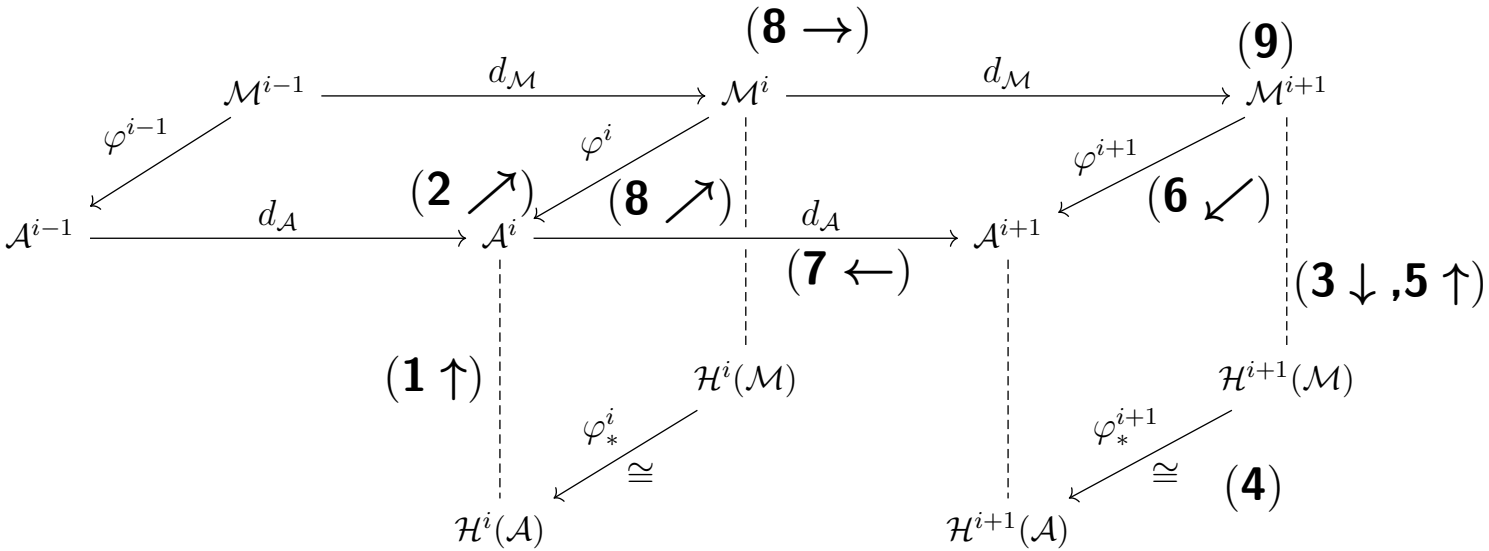
Effective computation of i-minimal models

MMarco's method

[Manero & Marco, 2020]

MMarco's method

Given a finitely presented CDGA $(\mathcal{A}, d_{\mathcal{A}})$, the effective method to compute its minimal model works as follows:



MMarco's method

Example (2.1 from [Manero & Marco, 2020])

Let $\mathcal{A}$ be a CDGA generated by six generators of degree 1: $e_1 \dots e_6$ and one generator of degree 2: e_7, with no relations and where the differential is given by:

$$d_{\mathcal{A}}(e_1) = -e_1 \wedge e_6$$

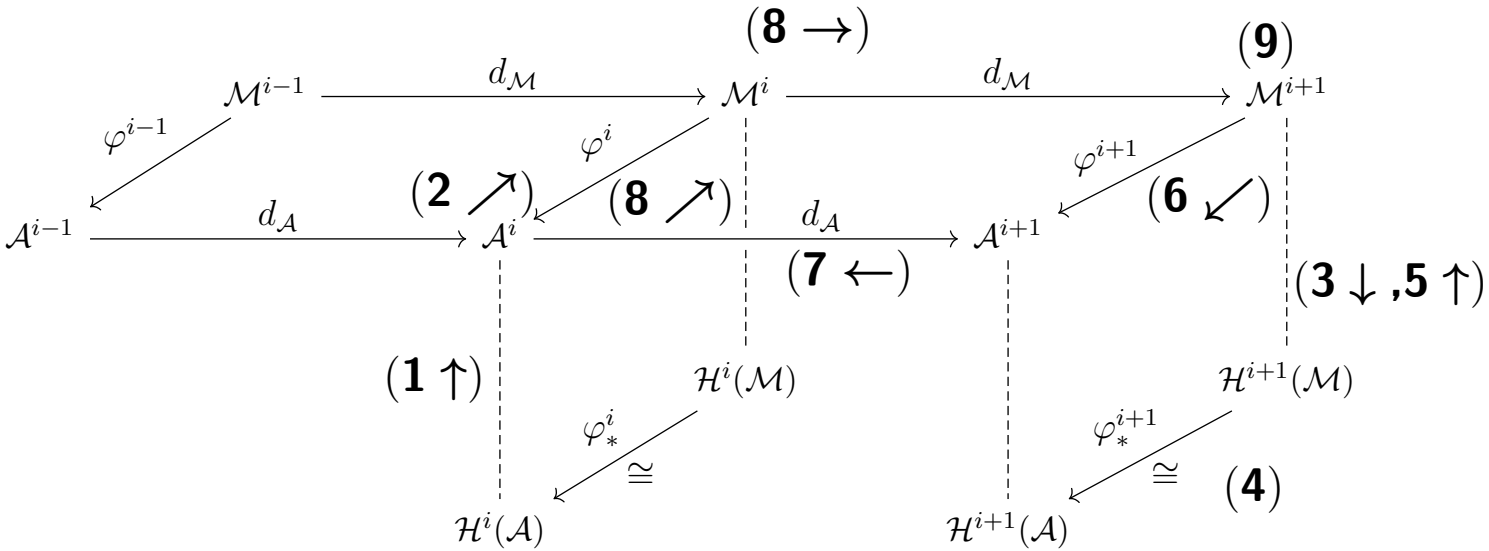
$$d_{\mathcal{A}}(e_2) = -e_2 \wedge e_6$$

$$d_{\mathcal{A}}(e_3) = -e_3 \wedge e_6$$

$$d_{\mathcal{A}}(e_4) = -e_5 \wedge e_6$$

$$d_{\mathcal{A}}(e_5) = d_{\mathcal{A}}(e_6) = d_{\mathcal{A}}(e_7) = 0$$

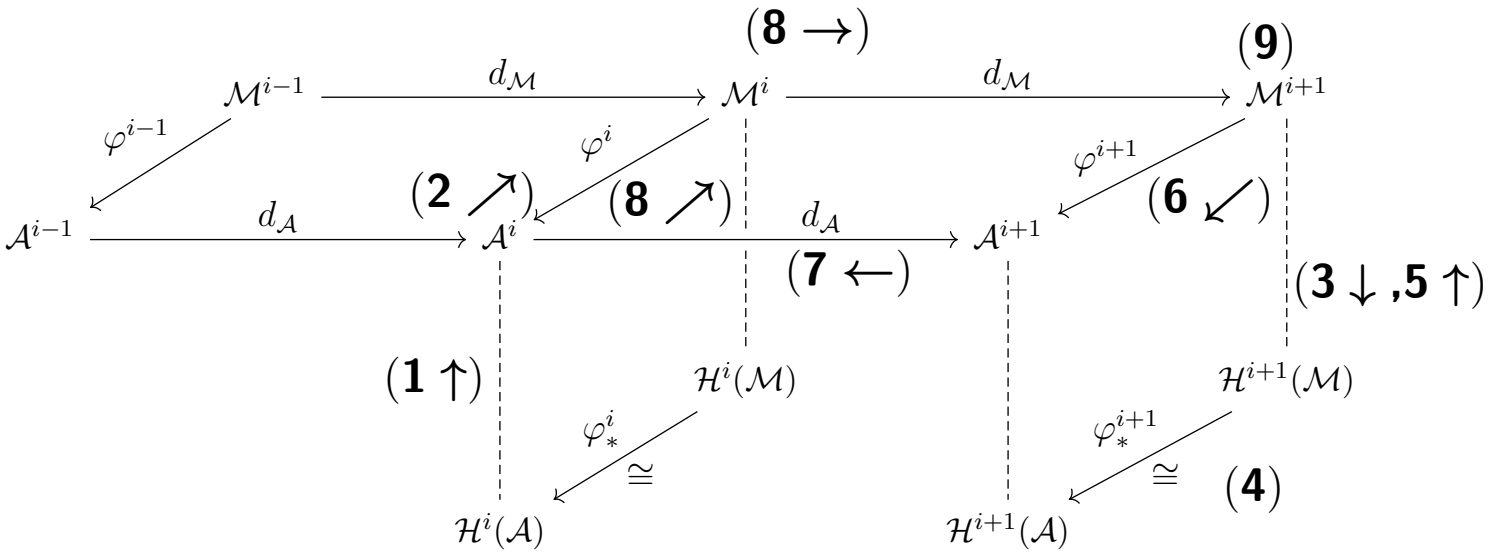
MMarco's method



Step 1: Get representatives of the first non-trivial cohomology group generators

$$\mathcal{H}^1(\mathcal{A}) = \mathbb{Q}\langle [e_5],\ [e_6] \rangle$$

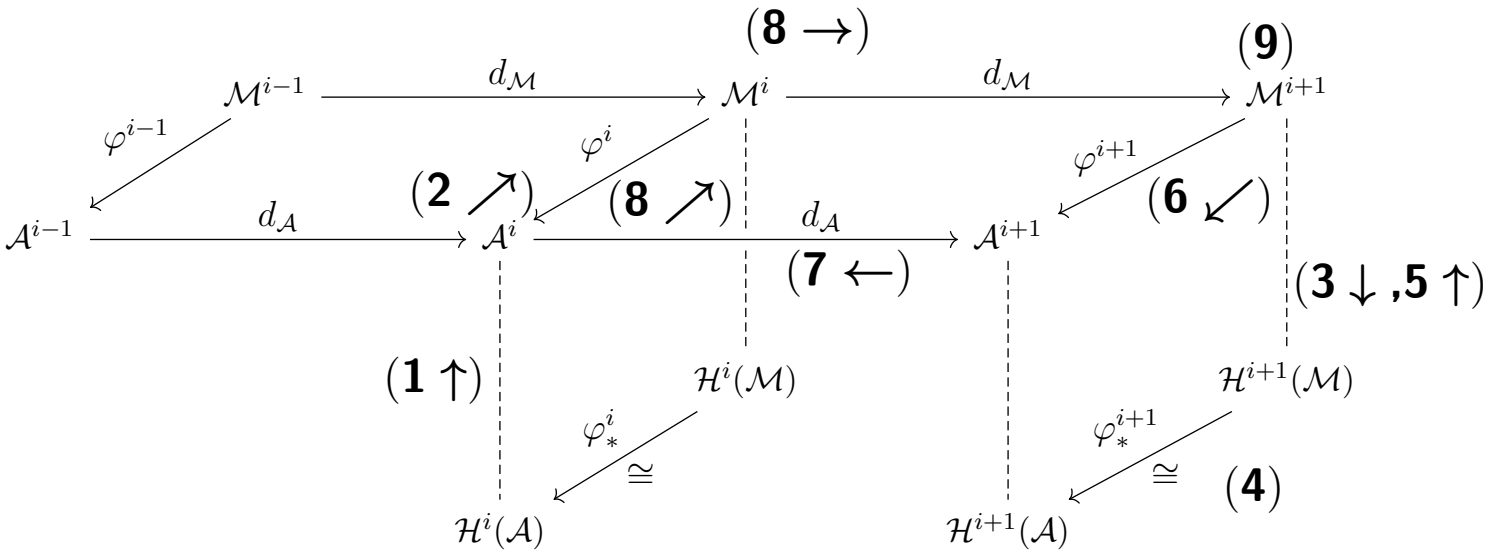
MMarco's method



Step 2: Update the model $(\mathcal{M}, d_{\mathcal{M}}, \varphi)$

$$\begin{array}{cccc} & \varphi & \mathcal{M} & d_{\mathcal{M}} \\ e_5 & \leftarrow & x_0^1 & \rightarrow \quad 0 \\ \\ e_6 & \leftarrow & x_1^1 & \rightarrow \quad 0 \end{array}$$

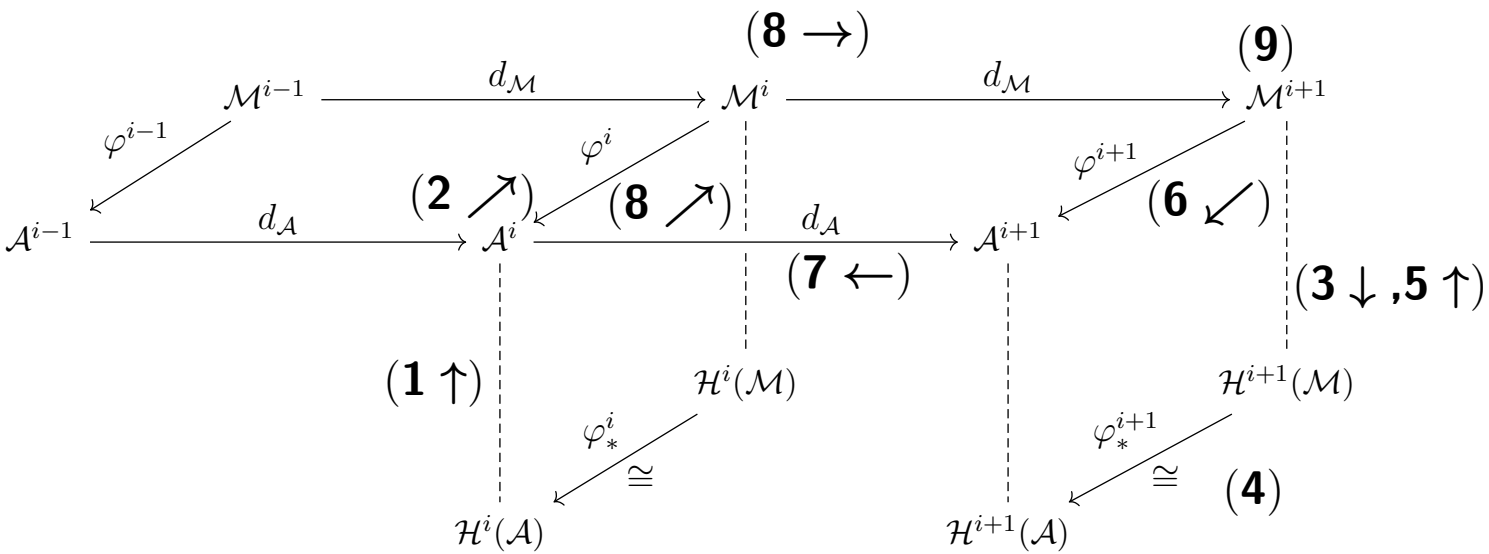
MMarco's method



Step 3: Compute $\mathcal{H}^2(\mathcal{M})$

$\mathcal{H}^2(\mathcal{M})$ is generated by the cohomology class $[x_0^1 \wedge x_1^1]$

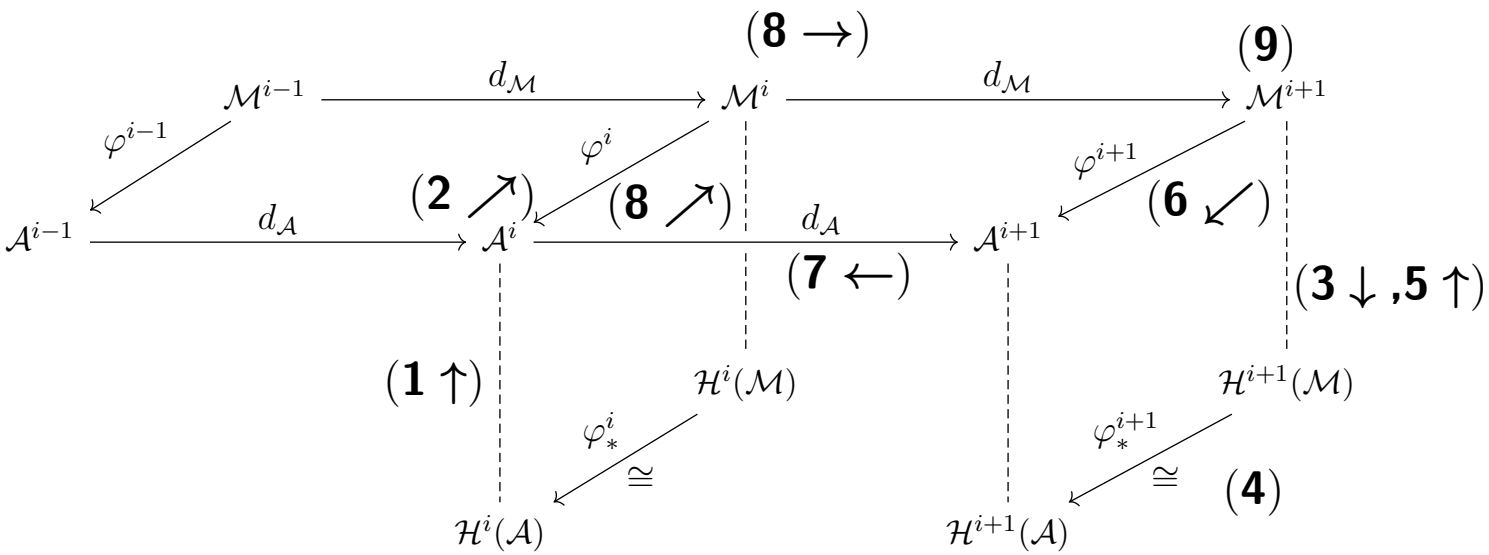
MMarco's method



Step 4: Compute $\ker(\varphi^2_*)$

$$\varphi^2_*([x^1_0 \wedge x^1_1]) = 0,$$

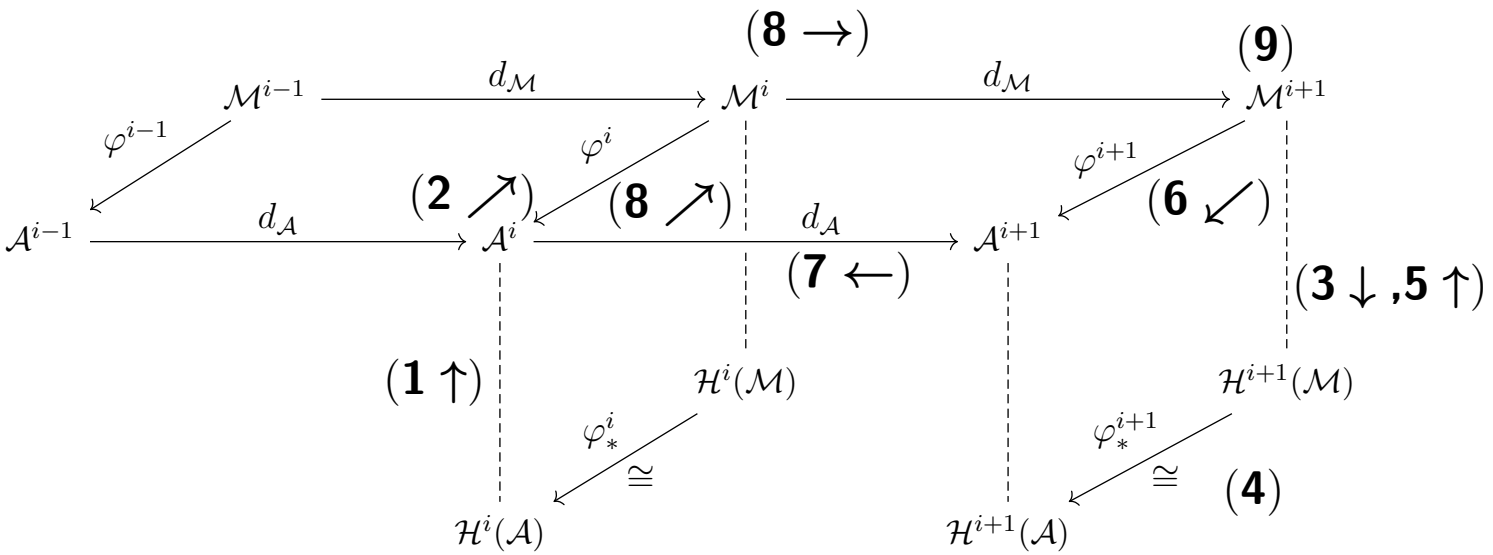
MMarco's method



Step 5: Get representatives of $\ker(\varphi^2_*)$

$$z_0^2 := x_0^1 \wedge x_1^1$$

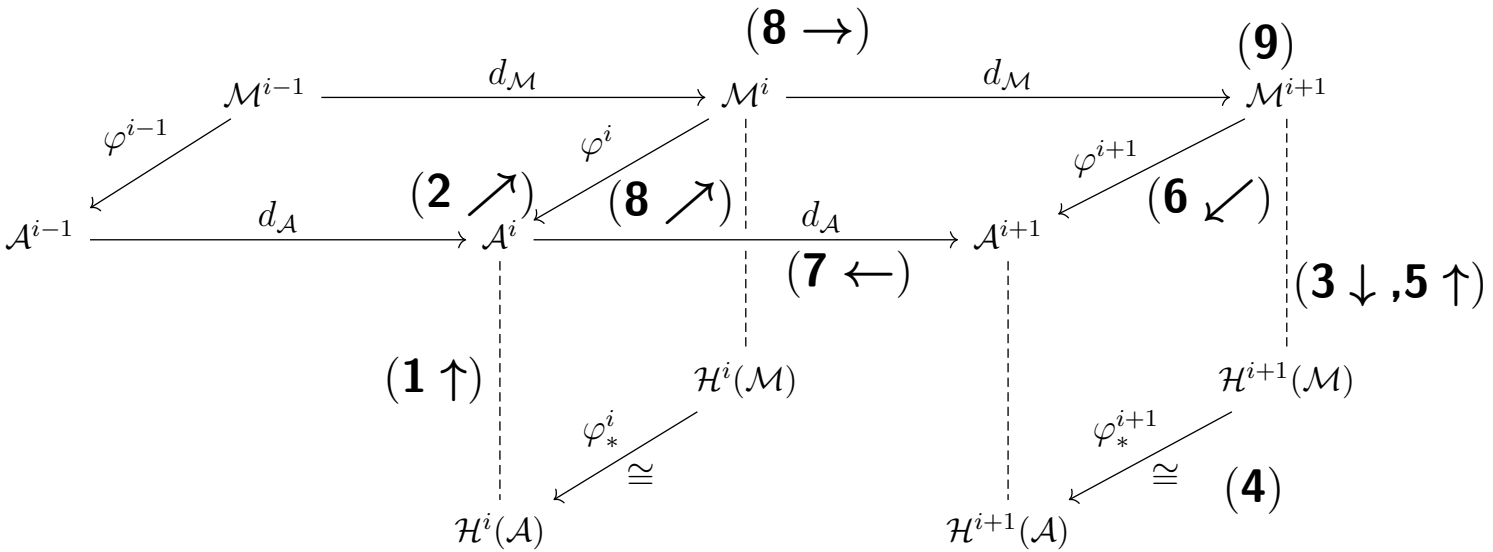
MMarco's method



Step 6: Get $\varphi(z_0^2)$

$$\varphi(z_0^2) = -e_5 \wedge e_6.$$

MMarco's method



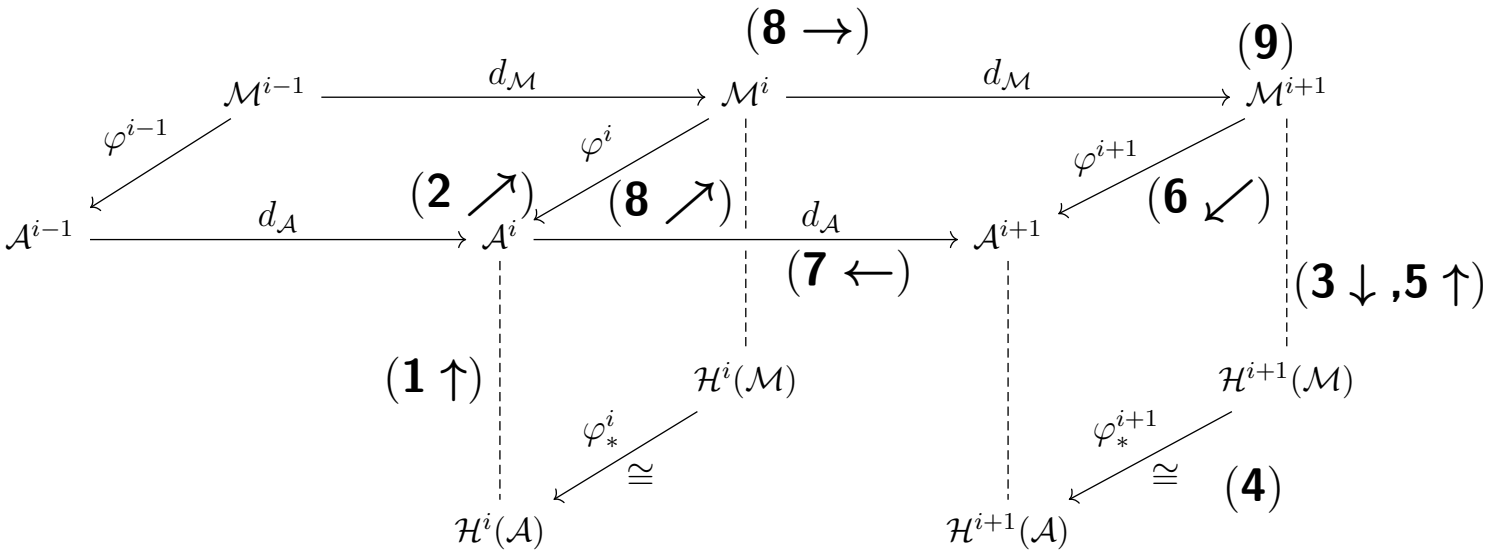
Step 7: Find $\theta \in \mathcal{A}^1$ s.t $d_{\mathcal{A}}(\theta) = \varphi(z_0^2)$

Solve the linear system:

$$d(\theta) = -e_5 \wedge e_6.$$

Solution: $\theta = e_4$.

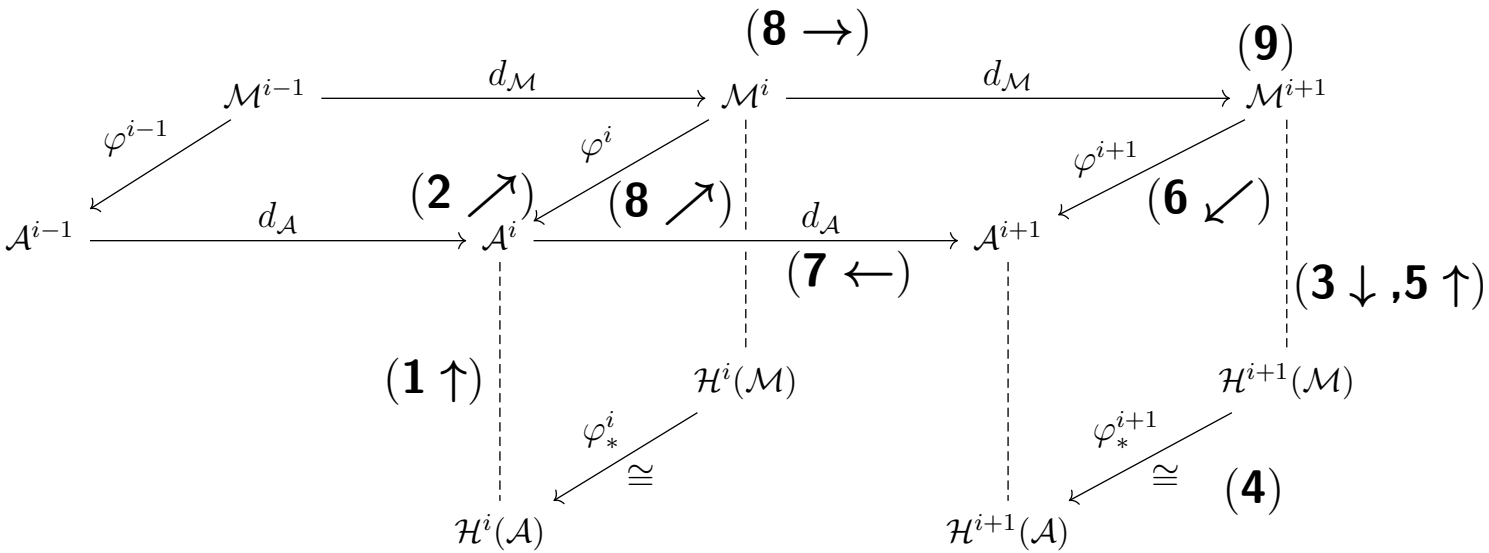
MMarco's method



Step 8: Update the model $(\mathcal{M}, d_{\mathcal{M}}, \varphi)$

	φ	$\mathcal{M}$	$d_{\mathcal{M}}$	
e_5	$\leftarrow$	x_0^1	$\rightarrow$	0
e_6	$\leftarrow$	x_1^1	$\rightarrow$	0
e_4	$\leftarrow$	y_0^1	$\rightarrow$	$x_0^1 \wedge x_1^1$

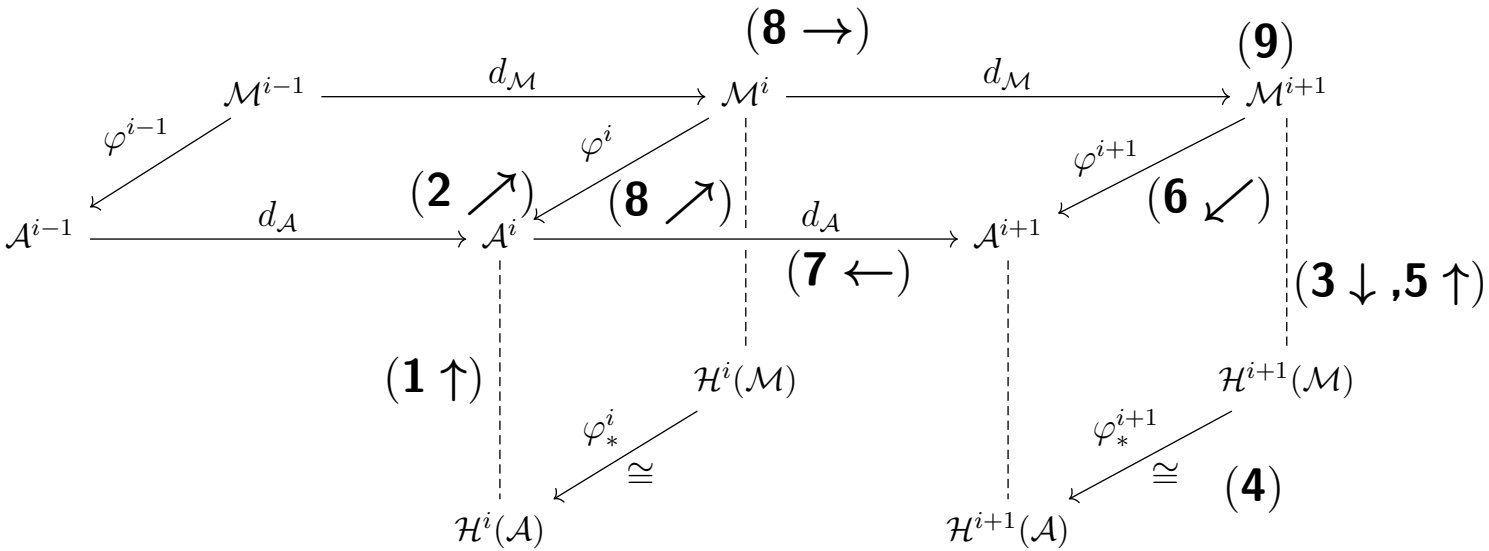
MMarco's method



Repeat steps 3 and 4 to check if φ^2_* is injective

In this case, the induced morphism is injective because $\varphi(x^1_0 \wedge y^1_0) = e_6 \wedge e_4$ and $\varphi(x^1_1 \wedge y^1_0) = e_5 \wedge e_4$ are representatives of different non-trivial cohomology classes that were not mapped before.

MMarco's method



Step 9: We have the 1-minimal model of $(\mathcal{A}, d_{\mathcal{A}})$

To continue, repeat all steps for degree $(i+1)$ to obtain an $(i+1)$-minimal model.

The minimal model of a topological space $\mathcal{X}$

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$

MMarco's method applied to topological spaces

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$

Every topological space $\mathcal{X}$ has associated a CDGA called the *algebra of polynomial differential forms on* $\mathcal{X}$. *This algebra is denoted by* $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$.

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Definition

The Sullivan minimal model of a topological space $\mathcal{X}$ is defined as the Sullivan minimal model of $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$.

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The core problem

MMarco's method assumes finitely presented homogeneous parts. However, in the case of the algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$, the homogeneous parts are **infinite-dimensional**.

Before defining $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$, a brief introduction to...

Simplicial sets

Simplicial objects

Definition

A simplicial object $\mathcal{K}$ with values in a category $\mathfrak{C}$ is a sequence $\{\mathcal{K}_n\}_{n\geq 0}$ of objects in $\mathfrak{C}$, together with $\mathfrak{C}$-morphisms:

$$\partial_i : \mathcal{K}_{n+1} \to \mathcal{K}_n,\ 0 \leq i \leq n+1, \qquad s_j : \mathcal{K}_n \to \mathcal{K}_{n+1},\ 0 \leq j \leq n,$$

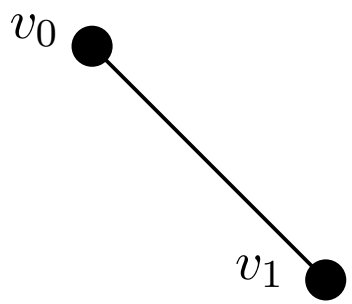
that satisfy the identities:

$$\partial_i \partial_j = \partial_{j-1} \partial_i,\ i < j$$

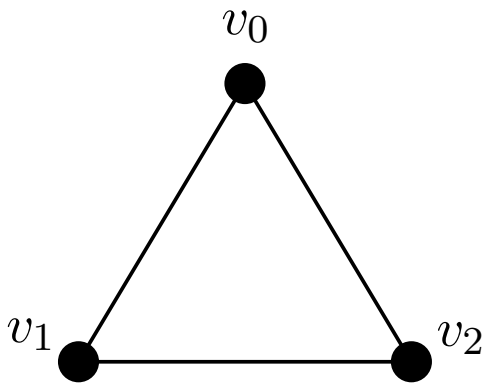
$$s_i s_j = s_{j+1} s_i,\ i \leq j$$

$$\partial_i s_j = \begin{cases} s_{j-1}\partial_i, & i < j \\ id, & i = j, j+1 \\ s_j \partial_{i-1}, & i > j+1 \end{cases}$$

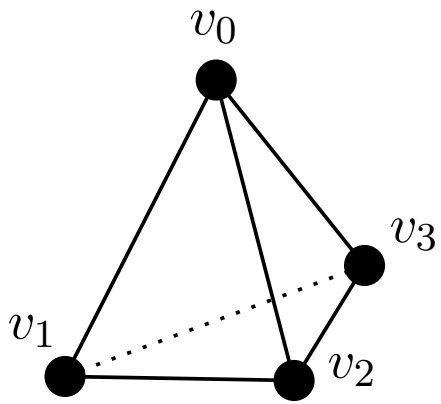
Abstract simplices



$[v_0, v_1]$



$[v_0, v_1, v_2]$



$[v_0, v_1, v_2, v_3]$

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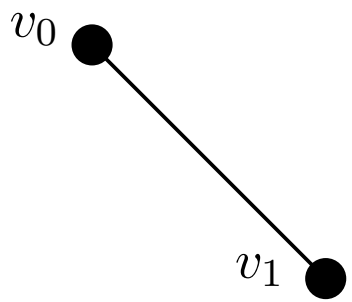

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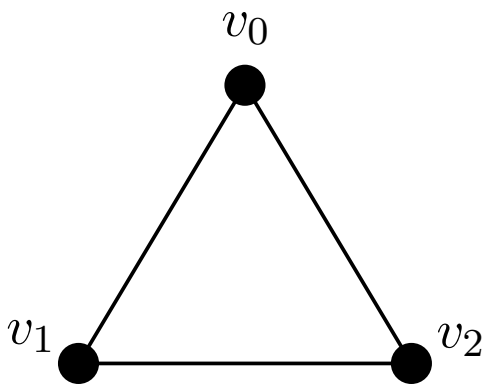

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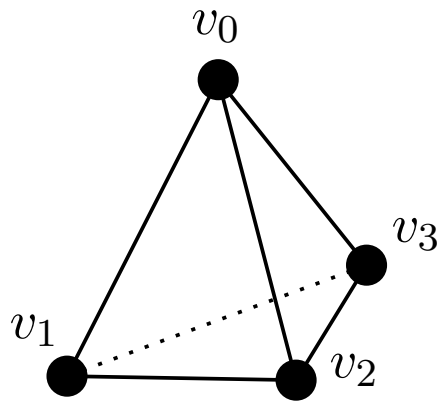
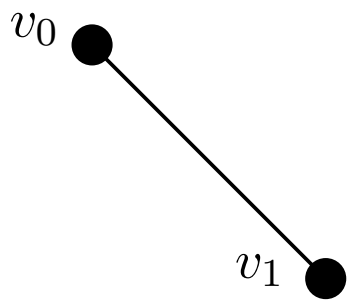


$[v_0, v_1, v_2, v_3]$

Face maps

$$\partial_i([v_0, \ldots, v_n]) = [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

Abstract simplices



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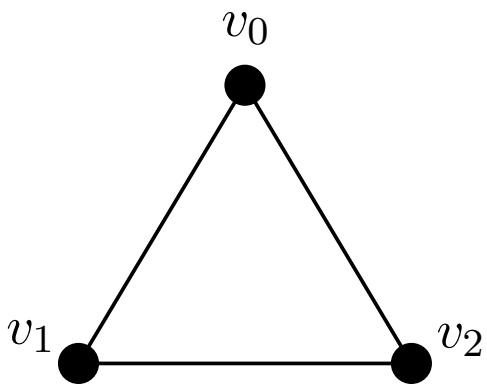
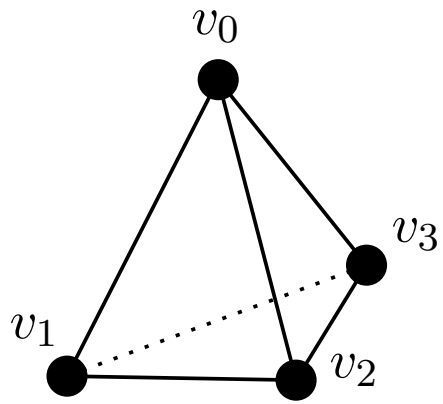


$[v_0, v_1, v_2]$

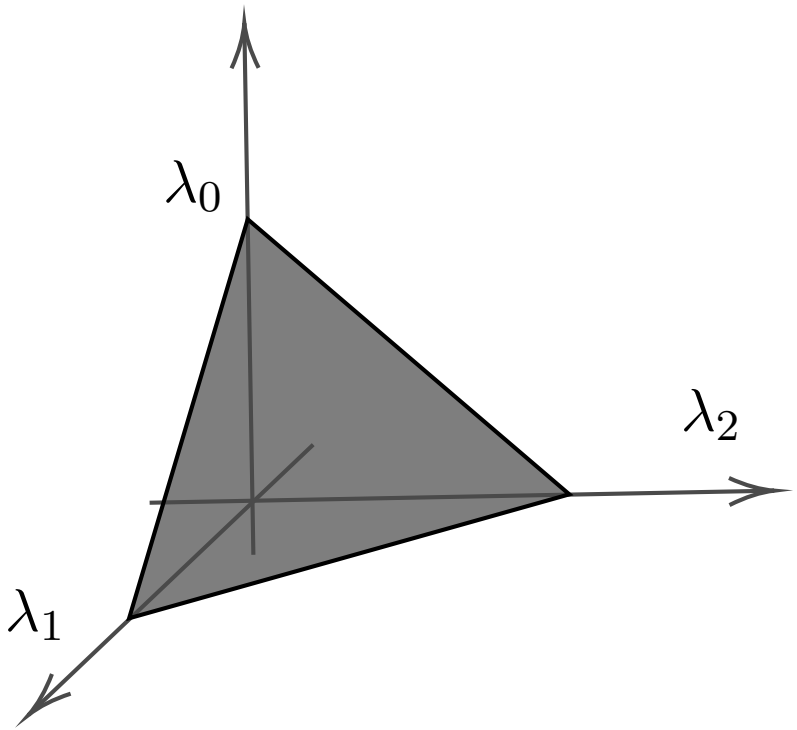


$[v_0, v_1, v_2, v_3]$

Degeneracy maps

$$s_i([v_0, \ldots, v_n]) = [v_0, v_1, \ldots, v_{i-1}, v_i, v_i, v_{i+1}, \ldots, v_n]$$

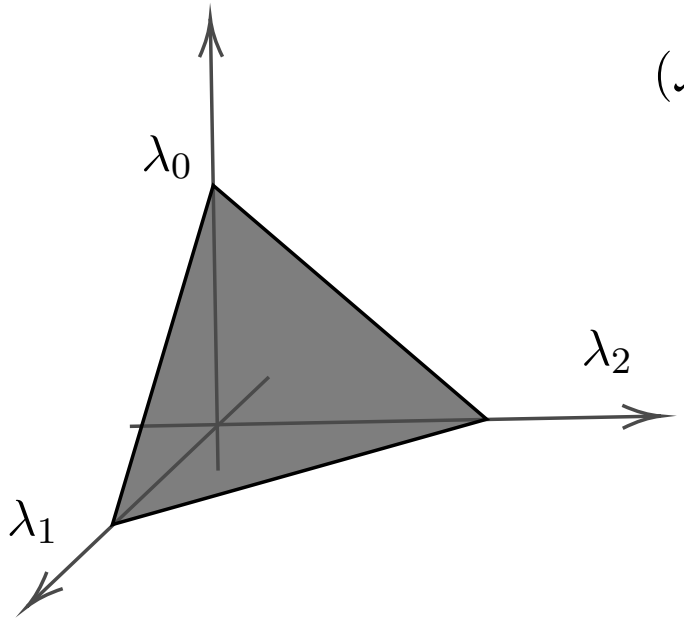
Geometrical simplex Δ^n



$$\Delta^n = \left\{(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} |\ \lambda_i \geq 0, \sum \lambda_i = 1 \right\}$$

The algebra of polynomial differential forms

Considering the algebra of polynomial differential forms on Δ^n, we define the cochain algebra $(\mathcal{A}_{\mathcal{PL}})_n$ as:



$$(\mathcal{A}_{\mathcal{PL}})_n = \frac{\bigwedge(t_0,\ldots,t_n,y_0,\ldots,y_n)}{\langle \sum t_i - 1, \sum y_j \rangle}$$

degrees

$$|t_i| = 0, \quad |y_j| = 1$$

differential

$$d(t_i) = y_i$$

The simplicial cochain algebra $\mathcal{A}_{\mathcal{PL}}$

We define the simplicial cochain algebra

$$\mathcal{A}_{\mathcal{PL}} = \{(\mathcal{A}_{\mathcal{PL}})_n\}_{n\geq 0},$$

where the face and degeneracy morphisms are the cochain algebra morphisms:

$$\partial_i : (\mathcal{A}_{\mathcal{PL}})_{n+1} \to (\mathcal{A}_{\mathcal{PL}})_n,\ 0 \leq i \leq n+1$$

$$s_j : (\mathcal{A}_{\mathcal{PL}})_n \to (\mathcal{A}_{\mathcal{PL}})_{n+1},\ 0 \leq j \leq n$$

given by

$$\partial_i : t_k \to \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases} \qquad s_j : t_k \to \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$

The CDGA we are interested in...

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$

Let $\mathcal{K}$ be a simplicial set, and let $\mathcal{A} = \{\mathcal{A}_n\}_{n\geq 0}$ be a simplicial cochain complex or a simplicial cochain algebra. Then

$$\mathcal{A}(\mathcal{K}) = \{\mathcal{A}^p(\mathcal{K})\}_{p\geq 0}$$

is the cochain complex (or cochain algebra) defined as follows:

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$$\mathcal{A}(\mathcal{K}) = \{\mathcal{A}^p(\mathcal{K})\}_{p\geq 0}$$

is the cochain complex (or cochain algebra) defined as follows:

• $\mathcal{A}^p(\mathcal{K})$ is the set of simplicial set morphisms from $\mathcal{K}$ to $\mathcal{A}^p$. Thus, an element $\Phi \in \mathcal{A}^p(\mathcal{K})$ is a mapping that assigns to each n-simplex $\sigma \in \mathcal{K}_n$ an element $\Phi_\sigma \in (\mathcal{A}^p)_n$, such that

$$\Phi_{\partial_i(\sigma)} = \partial_i(\Phi_\sigma)$$
$$\Phi_{s_j(\sigma)} = s_j(\Phi_\sigma)$$

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$

- Addition, scalar multiplication and the differential are given by

$$(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma$$

$$(\lambda\Phi)_\sigma = \lambda\Phi_\sigma$$

$$(d\Phi)_\sigma = d(\Phi_\sigma)$$

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$$(d\Phi)_\sigma = d(\Phi_\sigma)$$

- If $\mathcal{A}$ is a simplicial cochain algebra, multiplication in $\mathcal{A}(\mathcal{K})$ is given by

$$(\Phi\Psi)_\sigma = \Phi_\sigma\Psi_\sigma$$

The simplicial set of singular simplices $S(\mathcal{X})$

Let $\mathcal{X}$ be a topological space, and recall the definition of the standard simplex Δ^n:

$$\Delta^n = \left\{(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} |\ \lambda_i \geq 0,\ \sum \lambda_i = 1 \right\}.$$

Definition

The *simplicial set of singular simplices* is the simplicial set

$$S_*(\mathcal{X}) = \{S_n(\mathcal{X})\}_{n\geq 0},$$

where the set

$$S_n(\mathcal{X}) = \{\sigma : \Delta^n \to \mathcal{X}\}.$$

is called the set of singular n-simplices.

The simplicial set of singular simplices $S(\mathcal{X})$

Let $\mathcal{X}$ be a topological space, and recall the definition of the standard simplex Δ^n:

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Definition

Given a simplex $\sigma \in S_n(\mathcal{X})$ the face and degeneracy maps are defined as:

$$\partial_i(\sigma)(\lambda_0, \ldots, \lambda_{n-1}) = \sigma(\lambda_0, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_{n-1})$$

and

$$s_j(\sigma)(\lambda_0, \ldots, \lambda_{n+1}) = \sigma(\lambda_0, \ldots, \lambda_{j-1}, \lambda_j + \lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_{n+1})$$

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$

Let $\mathcal{X}$ be a topological space.

Definition

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$ is defined as the algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{S}(\mathcal{X}))$.

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The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{X})$ is defined as the algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{S}(\mathcal{X}))$.

The Sullivan minimal model of $\mathcal{X}$

The minimal model of the space $\mathcal{X}$ is the minimal model of the CDGA $\mathcal{A}_{\mathcal{PL}}(\mathcal{S}(\mathcal{X}))$.

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$ for a finite type simplicial set $\mathcal{K}$

Definition

A simplicial set $\mathcal{K}$ is said to be of **finite type** if it has a finite number of non-degenerated simplices.

The algebra $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$ for a finite type simplicial set $\mathcal{K}$

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A simplicial set $\mathcal{K}$ is said to be of **finite type** if it has a finite number of non-degenerated simplices.

Milnor realization functor $|\cdot|$

Every simplicial set $\mathcal{K}$ has associated a topological space $|\mathcal{K}|$ such that $|\mathcal{K}|$ and $|S(|\mathcal{K}|)|$ are homotopy equivalent, and

$$\mathcal{A}_{\mathcal{PL}}(S(|\mathcal{K}|)) \simeq \mathcal{A}_{\mathcal{PL}}(\mathcal{K})$$

are quasi-isomorphic.

1 Introduction

2 Adaptation points of MMarco's method

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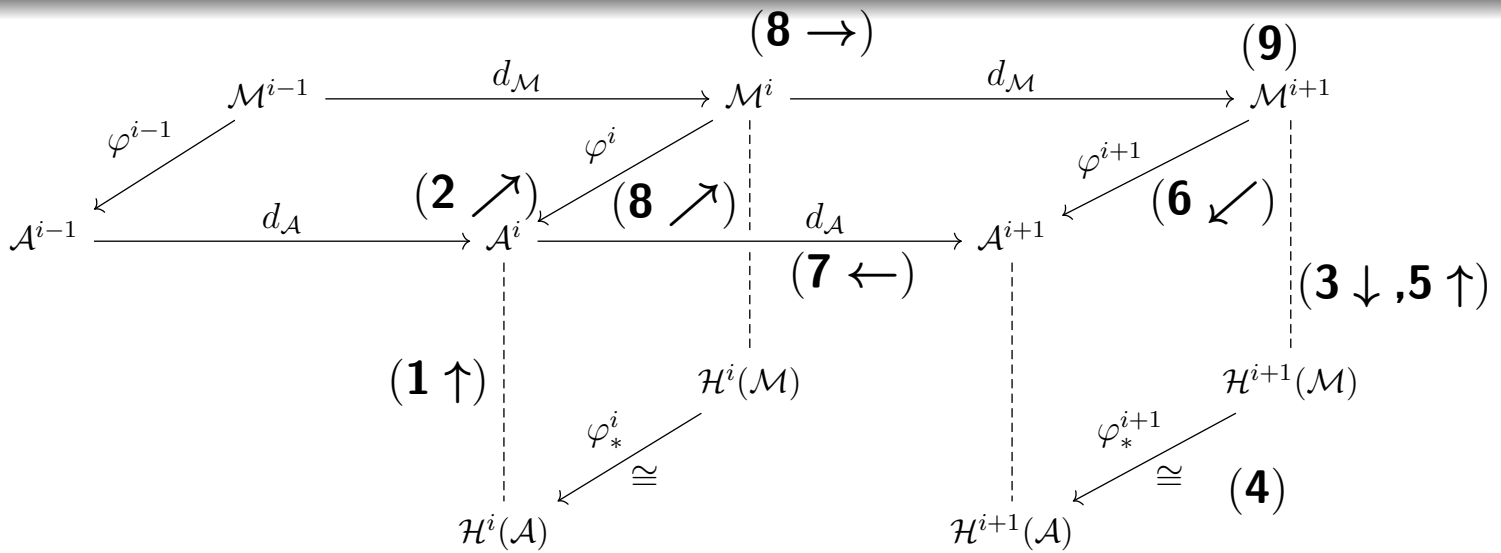
Identification of those steps where MMarco's method needs adaptations



(P1) Effective representation of the $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$ elements.

(P2) Inability to compute a basis of $\mathcal{H}^*(\mathcal{A}_{\mathcal{PL}}(\mathcal{K}))$ $\Rightarrow$ **[Steps 1, 3 and 4]**

(P3) Given an exact element $\varphi(z) \in \mathcal{A}^{i+1}_{\mathcal{PL}}(\mathcal{K})$, inability to find a primitive $\theta \in \mathcal{A}^{i}_{\mathcal{PL}}(\mathcal{K})$ such that $d_{\mathcal{A}}(\theta) = \varphi(z)$ $\Rightarrow$ **[Step 7]**.

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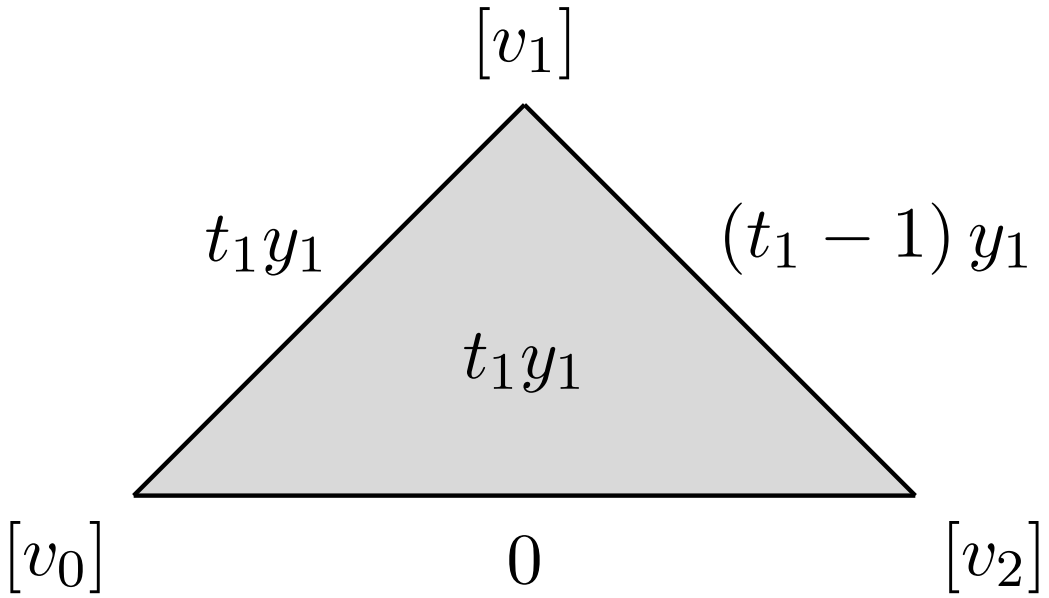
Representation of the elements of $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$



As a table:

$$
\begin{aligned}
Simplex\,(v_0, v_1) &\longrightarrow t_1 y_1 \\
Simplex\,(v_1, v_2) &\longrightarrow (t_1 - 1)\, y_1 \\
Simplex\,(v_0, v_1, v_2) &\longrightarrow t_1 y_1 \\
Otherwise &\longrightarrow 0
\end{aligned}
$$

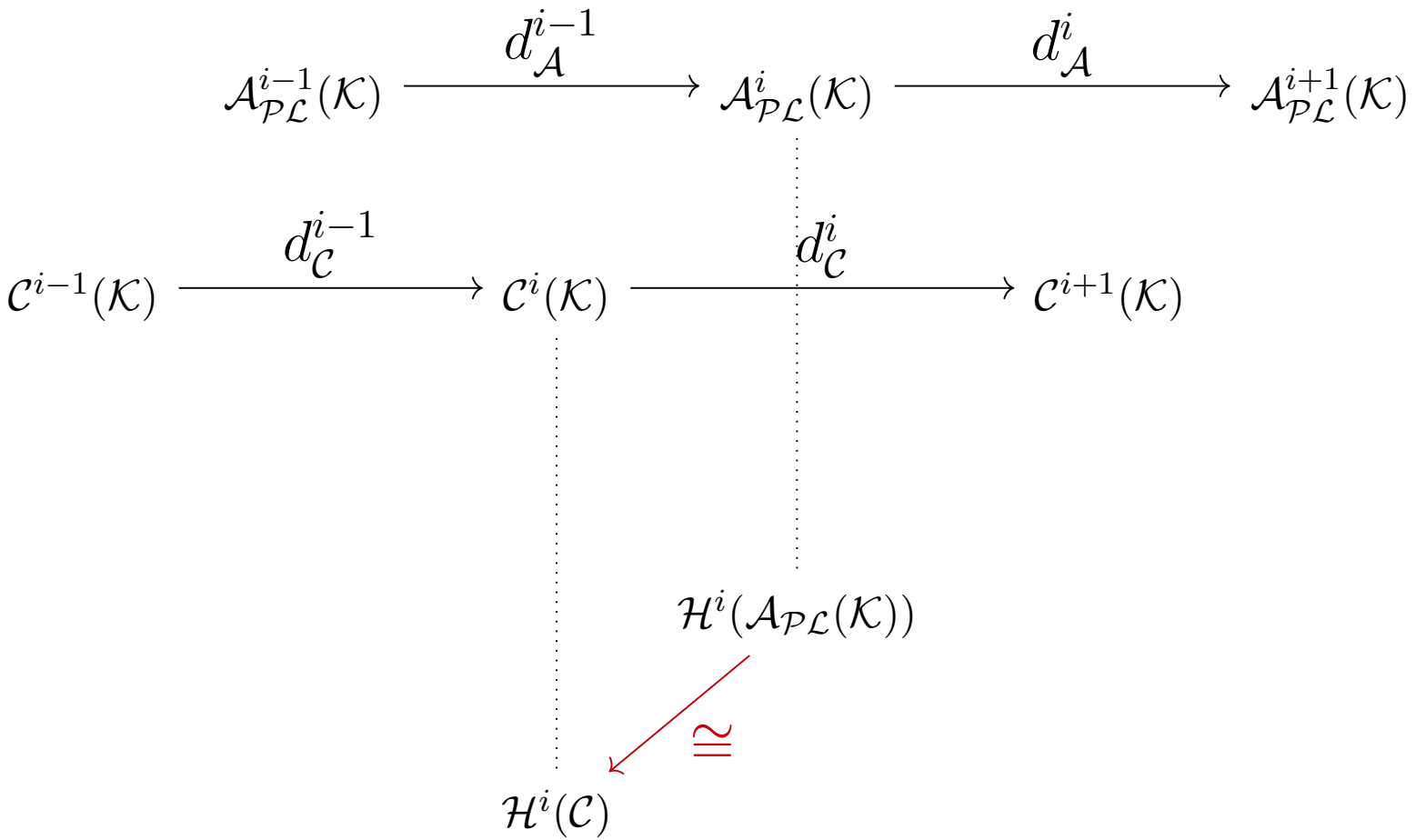
Step 1: Get harmonic representatives of $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$

$$\mathcal{A}^{i-1}_{\mathcal{PL}}(\mathcal{K}) \xrightarrow{d^{i-1}_{\mathcal{A}}} \mathcal{A}^{i}_{\mathcal{PL}}(\mathcal{K}) \xrightarrow{d^{i}_{\mathcal{A}}} \mathcal{A}^{i+1}_{\mathcal{PL}}(\mathcal{K})$$

$$\frac{\ker(d^i_{\mathcal{A}})}{Im(d^{i-1}_{\mathcal{A}})}$$

$$\mathcal{H}^i(\mathcal{A}_{\mathcal{PL}}(\mathcal{K}))$$

Alternative approach: $\mathcal{C}(\mathcal{K})$ as a proxy to $\mathcal{H}(\mathcal{A}_{\mathcal{PL}}(\mathcal{K}))$



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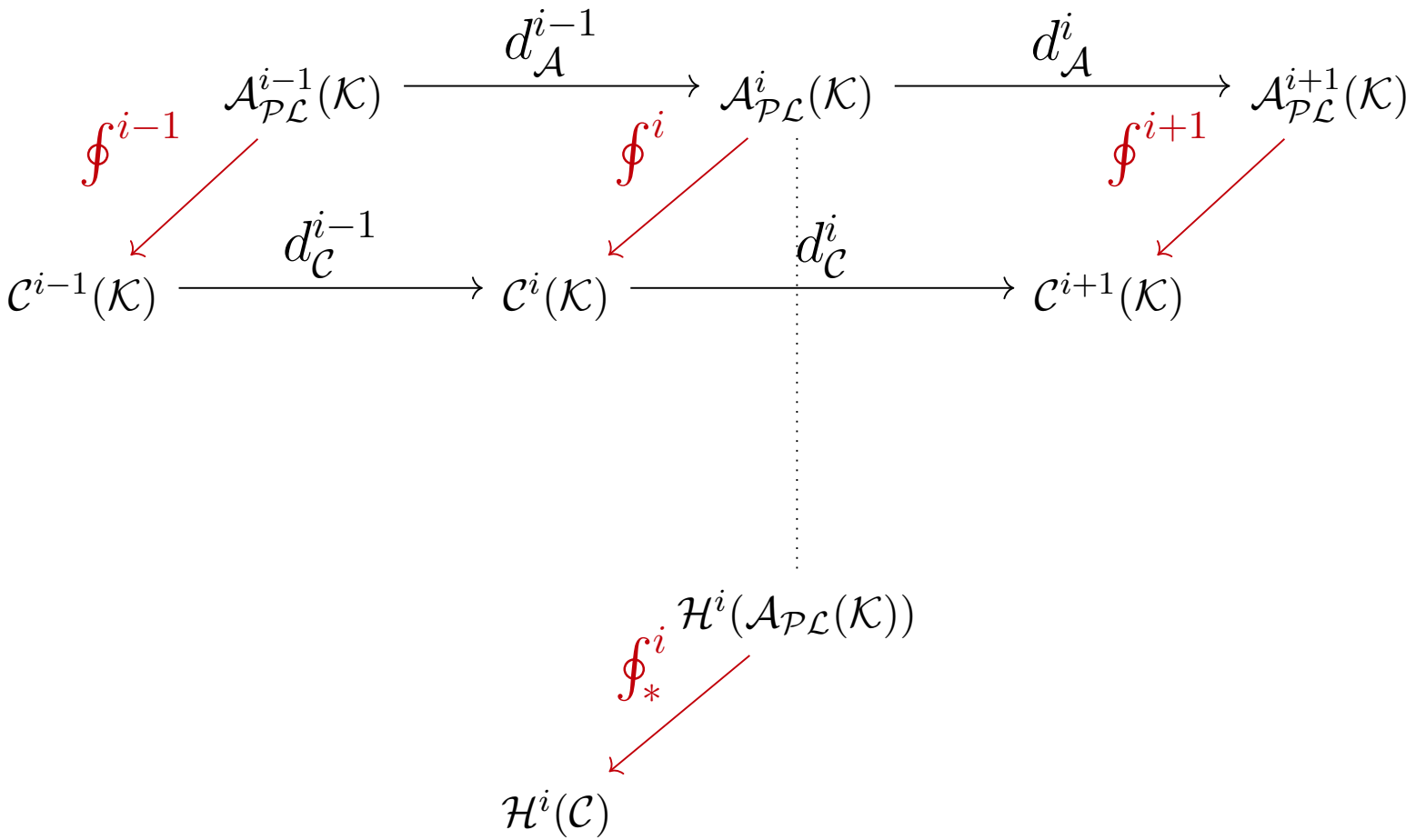


The de Rham integration map

$$\oint : \mathcal{A}_{\mathcal{PL}}(\mathcal{K}) \to \mathcal{C}(\mathcal{K})$$

The integration map $\oint$

Let $\int_n$ be the linear map:

$$\int_n : (\mathcal{A}_{\mathcal{PL}})^n_n \to \mathbb{Q}$$

by setting

$$\int_n t_1^{k_1} \dots t_n^{k_n} dt_1 \wedge \dots \wedge dt_n = \int_0^1 dt_1 \int_0^{1-t_1} dt_2 \dots$$
$$\dots \int_0^{1-\sum_1^{n-1} t_i} t_1^{k_1} \dots t_n^{k_n} dt_n =$$
$$= \frac{k_1! k_2! \dots k_n!}{(k_1 + \dots + k_n + n)!}.$$

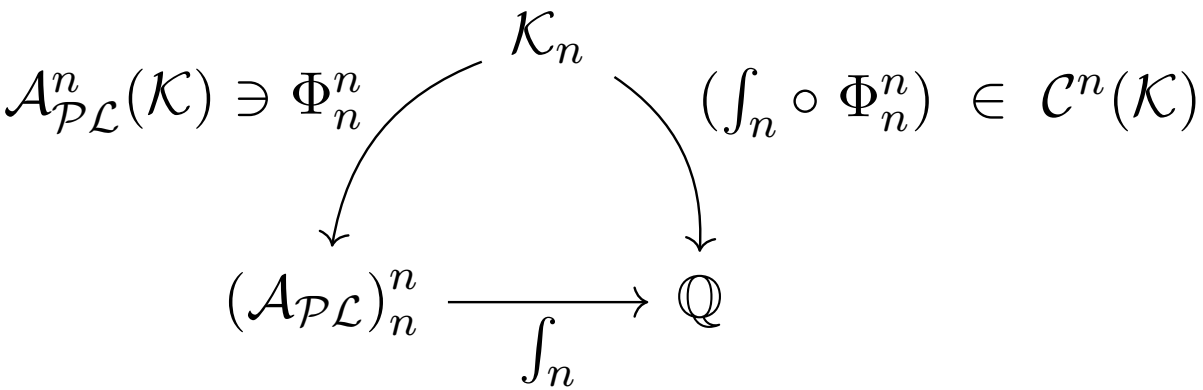
The integration map $\oint$

Using this map, we can integrate elements of $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$ as follows: consider a homogeneous element $\Phi^n \in \mathcal{A}^n_{\mathcal{PL}}(\mathcal{K})$, and $\Phi^n_m : \mathcal{K}_m \to (\mathcal{A}_{\mathcal{PL}})^n_m$ its restriction to simplicies of dimension m. We define the integral of the homogeneous element Φ^n as the map:

$$\begin{array}{rccc} \oint : & \mathcal{A}^n_{\mathcal{PL}}(\mathcal{K}) & \to & \mathcal{C}^n(\mathcal{K}) \\ & \Phi^n & \to & (\int_n \circ \Phi^n_n). \end{array}$$

The integration map $\oint$

The integral of the element Φ^n is the element of $\mathcal{C}^n(\mathcal{K})$ determined by the integration map $\int_n$ over the set of n-simplicies:



The AT-model of $\mathcal{C}(\mathcal{K})$

$$\cdots \quad \mathcal{C}^{i-1}(\mathcal{K}) \underset{\phi^i_{\mathcal{C}}}{\overset{d^{i-1}_{\mathcal{C}}}{\rightleftarrows}} \mathcal{C}^{i}(\mathcal{K}) \underset{\phi^{i+1}_{\mathcal{C}}}{\overset{d^{i}_{\mathcal{C}}}{\rightleftarrows}} \mathcal{C}^{i+1}(\mathcal{K}) \quad \cdots$$

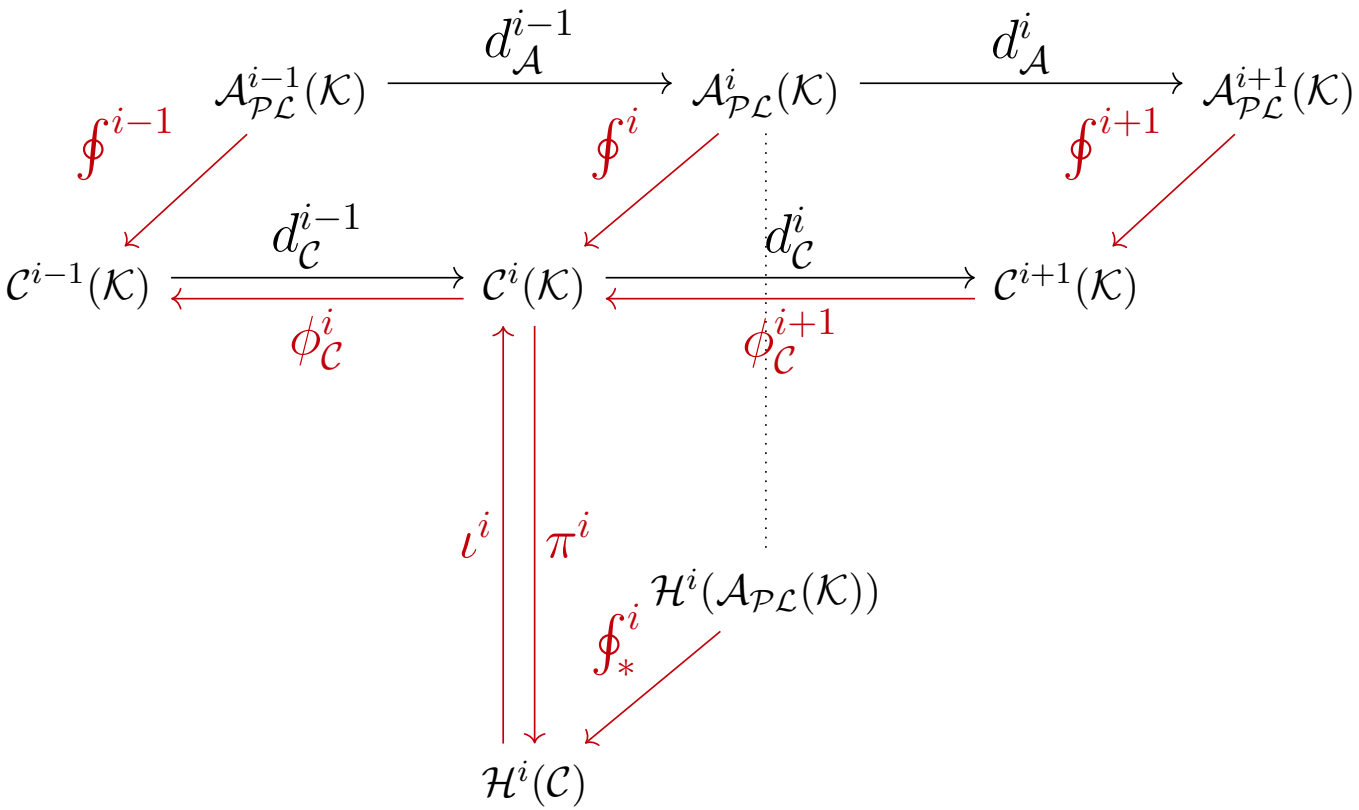
$$\iota_i \uparrow \downarrow \pi_i$$

$$\mathcal{H}_n$$

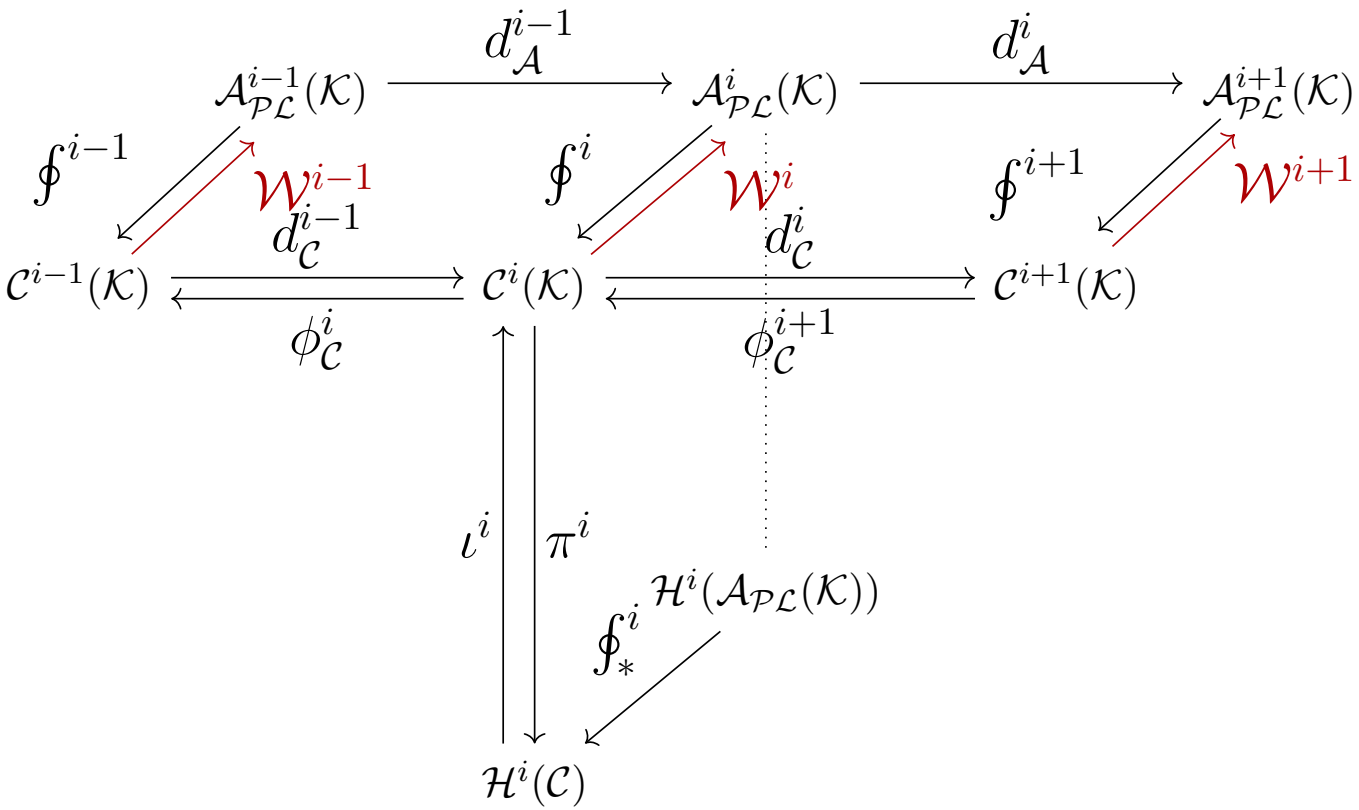
$$\mathcal{C}^n(\mathcal{K}) = \mathcal{H}^n \oplus \mathcal{X}^n \oplus \mathcal{Y}^n, \text{ where}$$

$$\begin{aligned} \mathcal{H}^n &:= Im(\iota \circ \pi), \\ \mathcal{X}^n &:= Im(d_{\mathcal{C}} \circ \phi), \\ \mathcal{Y}^n &:= Im(\phi \circ d_{\mathcal{C}}). \end{aligned}$$

Alternative approach: $\mathcal{C}(\mathcal{K})$ as a proxy to $\mathcal{H}(\mathcal{A}_{\mathcal{PL}}(\mathcal{K}))$



Step 2: Get harmonic representatives on $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$



The Whitney map $\mathcal{W}$

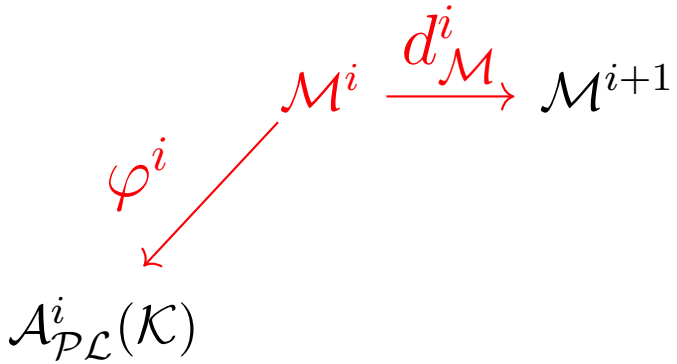
$$\mathcal{W} : \mathcal{C}(\mathcal{K}) \to \mathcal{A}_{\mathcal{PL}}(\mathcal{K})$$

$$\mathcal{W}^p(\beta)(\sigma_n) = \sum_{\gamma \in \overline{\partial}_p^n} \beta(\overline{\partial}_\gamma(\sigma_n)) \cdot \omega_n^p(\sigma_n, \gamma),$$

where

$$\omega_n^p(\sigma_n, \gamma) = (p!) \sum_{i=0}^{p} (-1)^i \cdot t_{j_i} \cdot y_{j_0} \wedge \cdots \wedge \hat{y}_{j_i} \wedge \cdots \wedge y_{j_p}.$$

Step 3: Update the model $(\mathcal{M}, d_{\mathcal{M}})$



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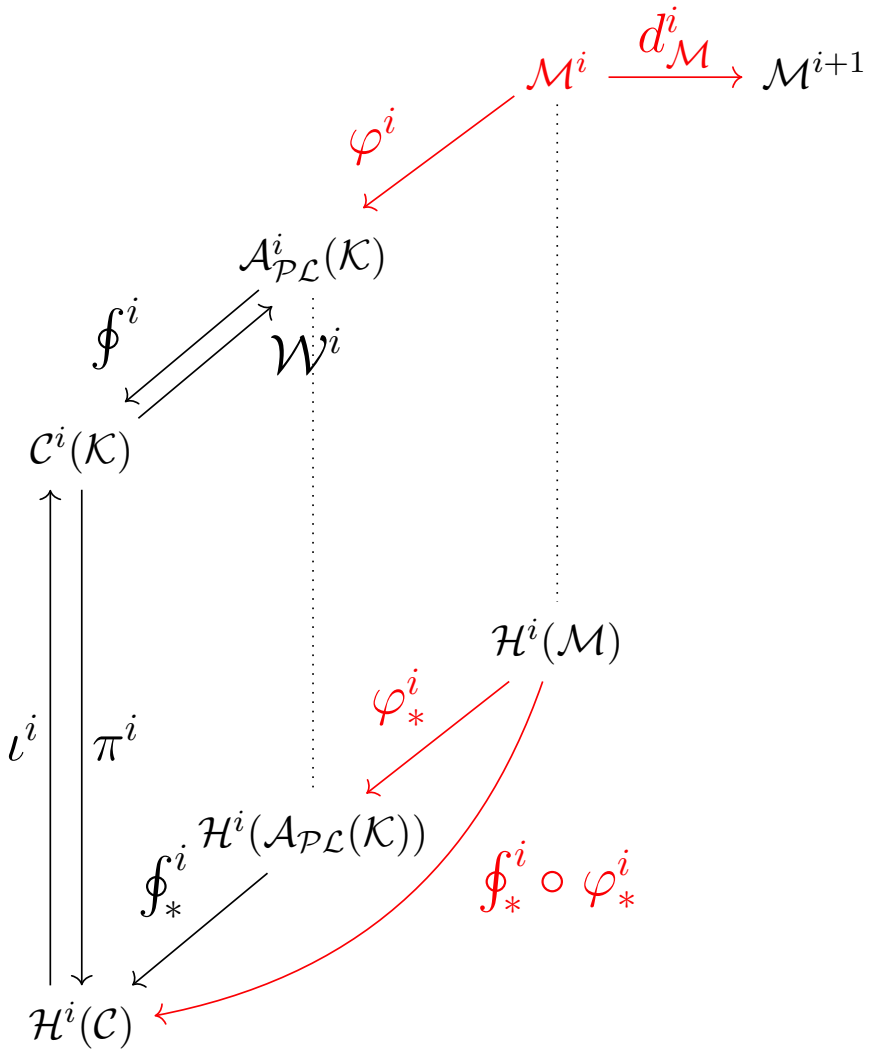
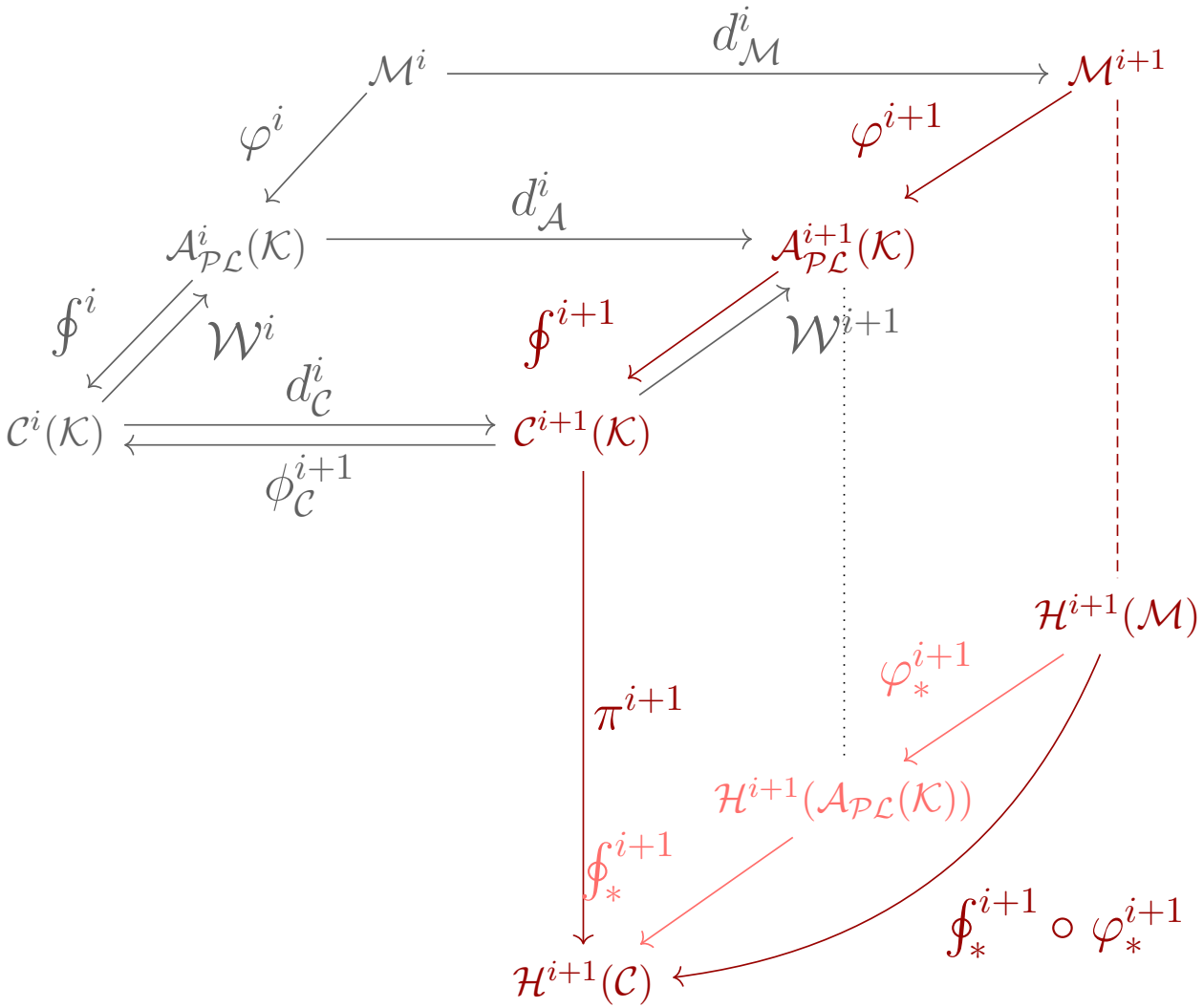


Step 4: Computation of $\mathcal{H}^{i+1}(\mathcal{M})$

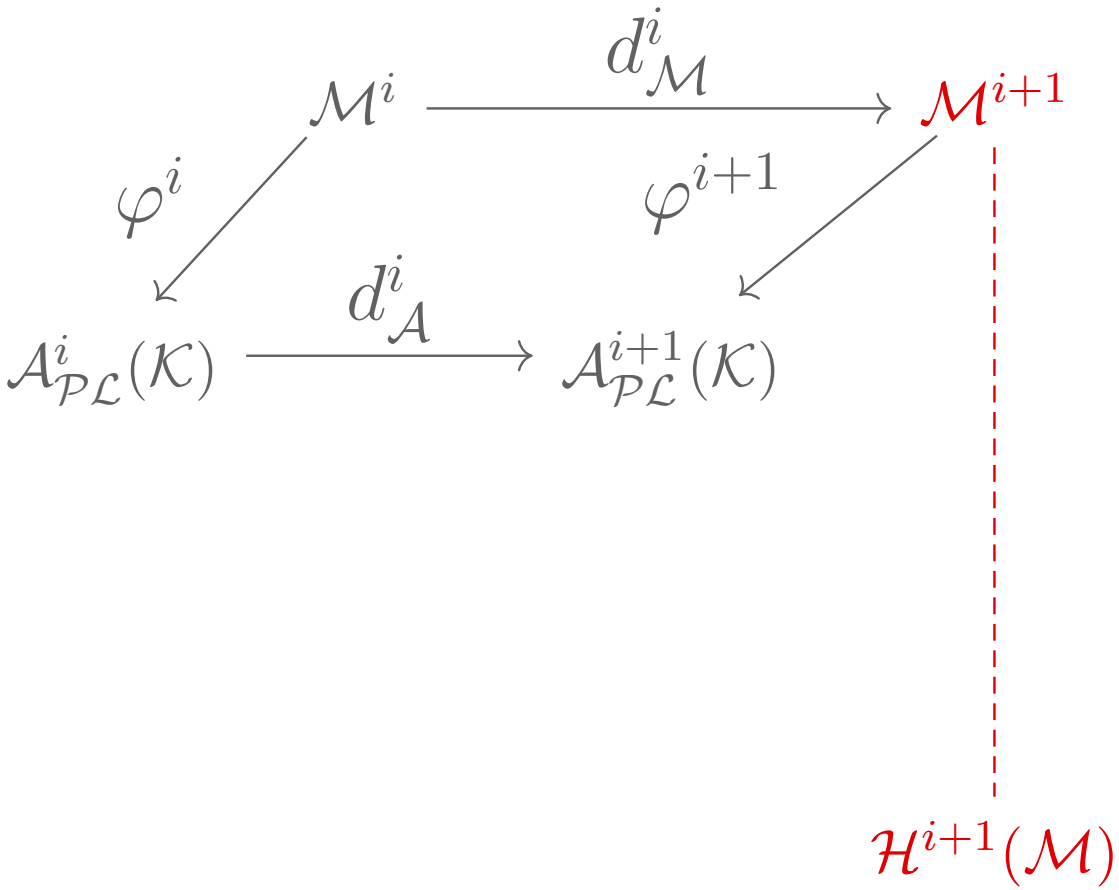
$$\mathcal{M}^i \xrightarrow{d^i_{\mathcal{M}}} \mathcal{M}^{i+1} \xrightarrow{d^{i+1}_{\mathcal{M}}} \mathcal{M}^{i+2}$$

$$\mathcal{H}^{i+1}(\mathcal{M})$$

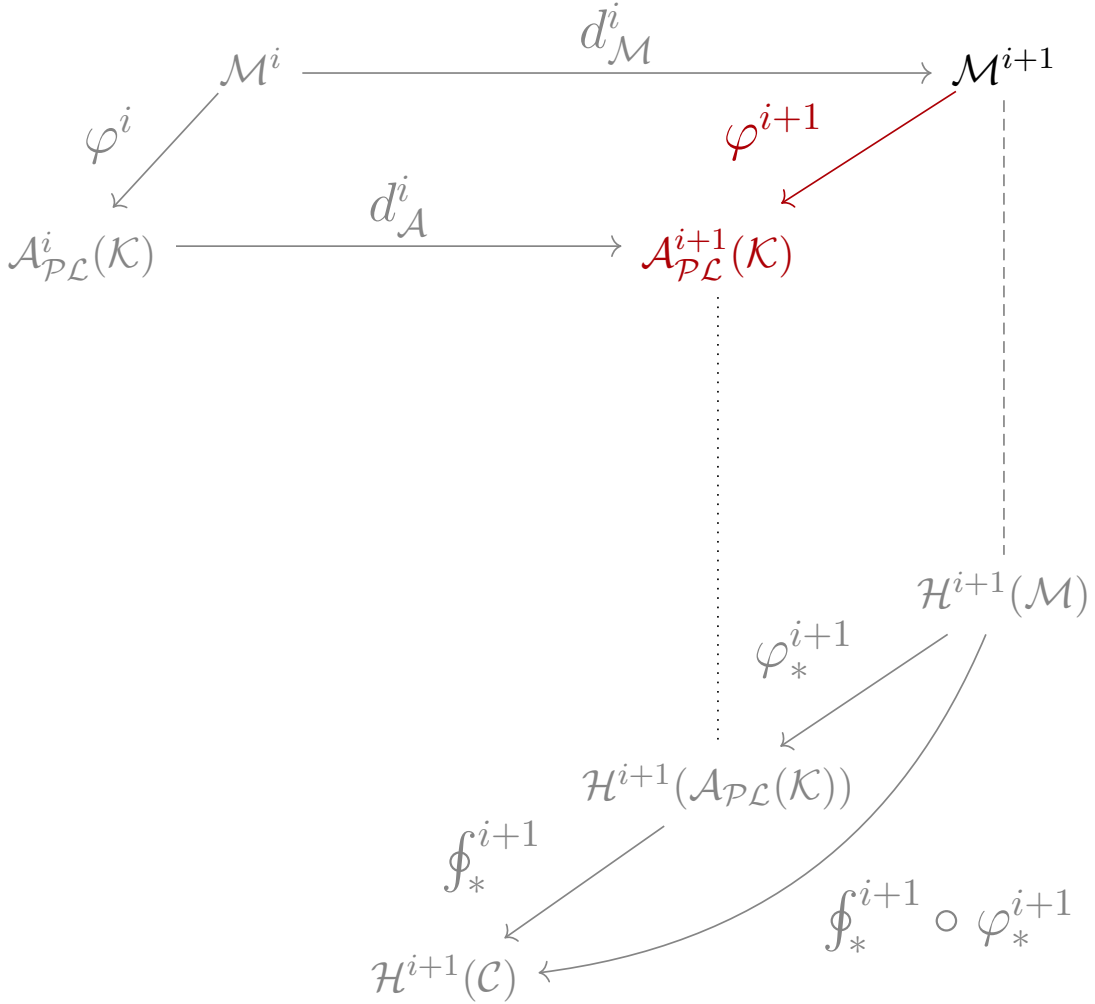
Step 5: Computation of $ker(\oint_*^{i+1} \circ \varphi_*^{i+1})$



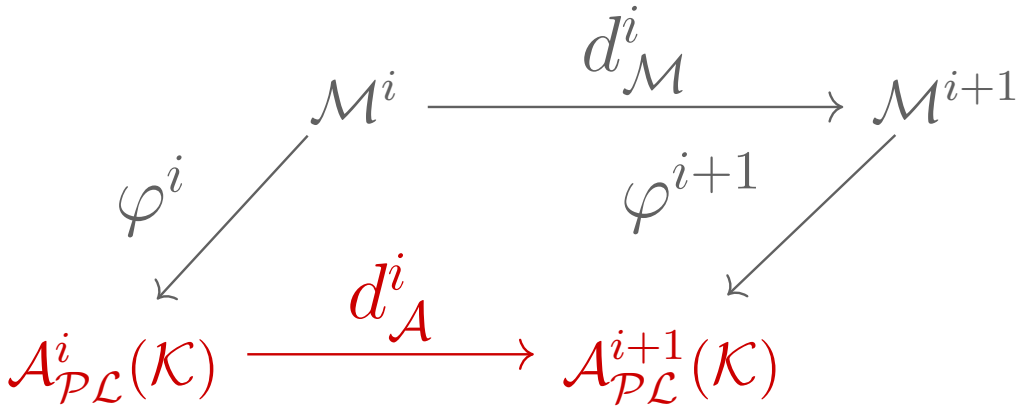
Step 6: Get representatives of $ker(\oint_*^{i+1} \circ \varphi_*^{i+1})$



Step 7: Apply φ to get representatives on $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$



Step 8: Find a preimage by $d^i_{\mathcal{A}}$ of the exact elements



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Method 1: naïf approach

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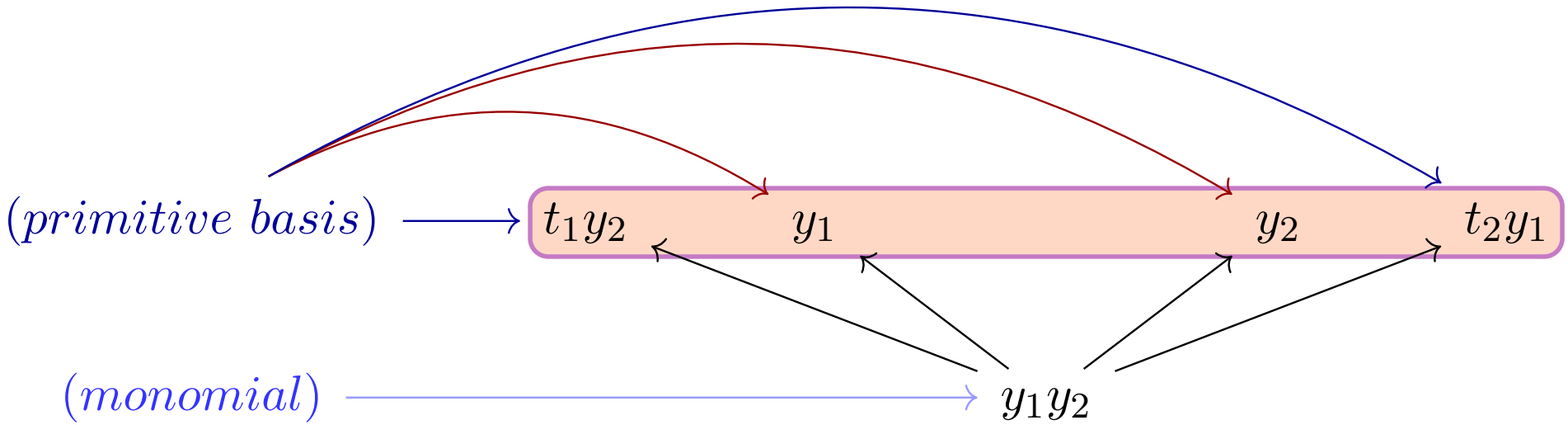
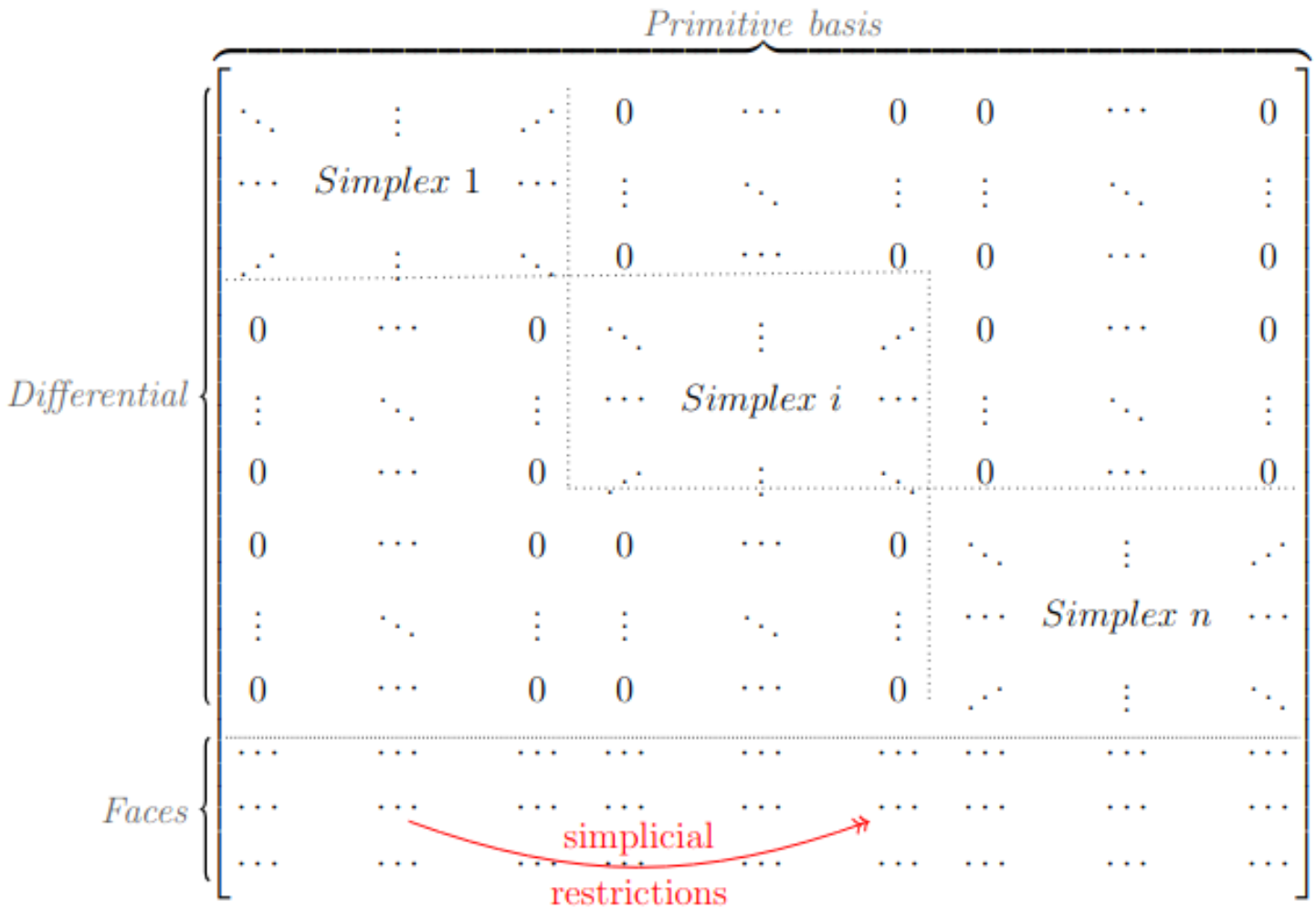


Method 1: naïf approach



Step 8: Find a preimage by $d^i_{\mathcal{A}}$ of the exact elements

Method 2: the combinatorial approach

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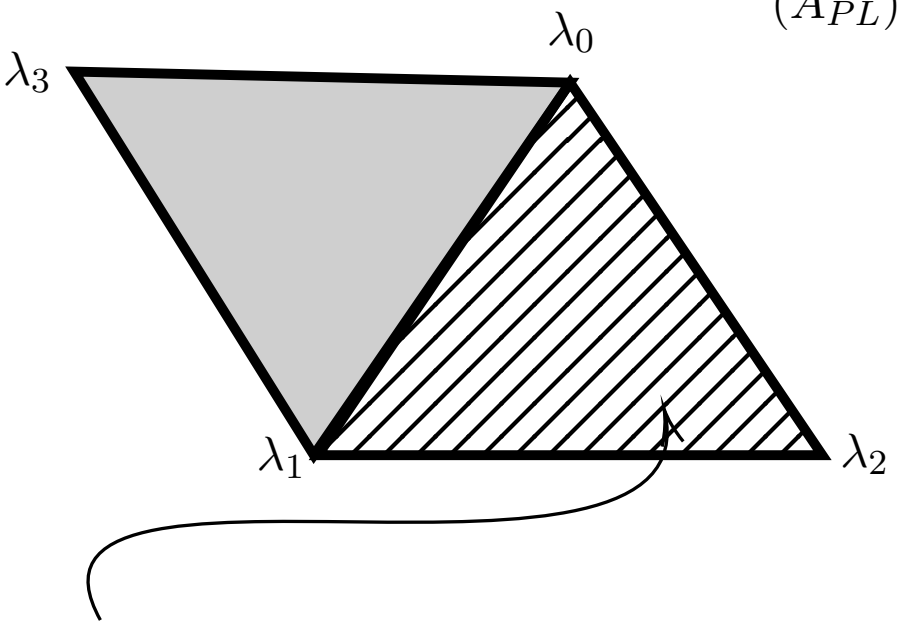
$$K_0 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$$

$$K_1 = \{(\lambda_0, \lambda_1), (\lambda_0, \lambda_2), (\lambda_1, \lambda_2), (\lambda_0, \lambda_3), (\lambda_1, \lambda_3)\}$$

$$K_2 = \{(\lambda_0, \lambda_1, \lambda_3)\}$$

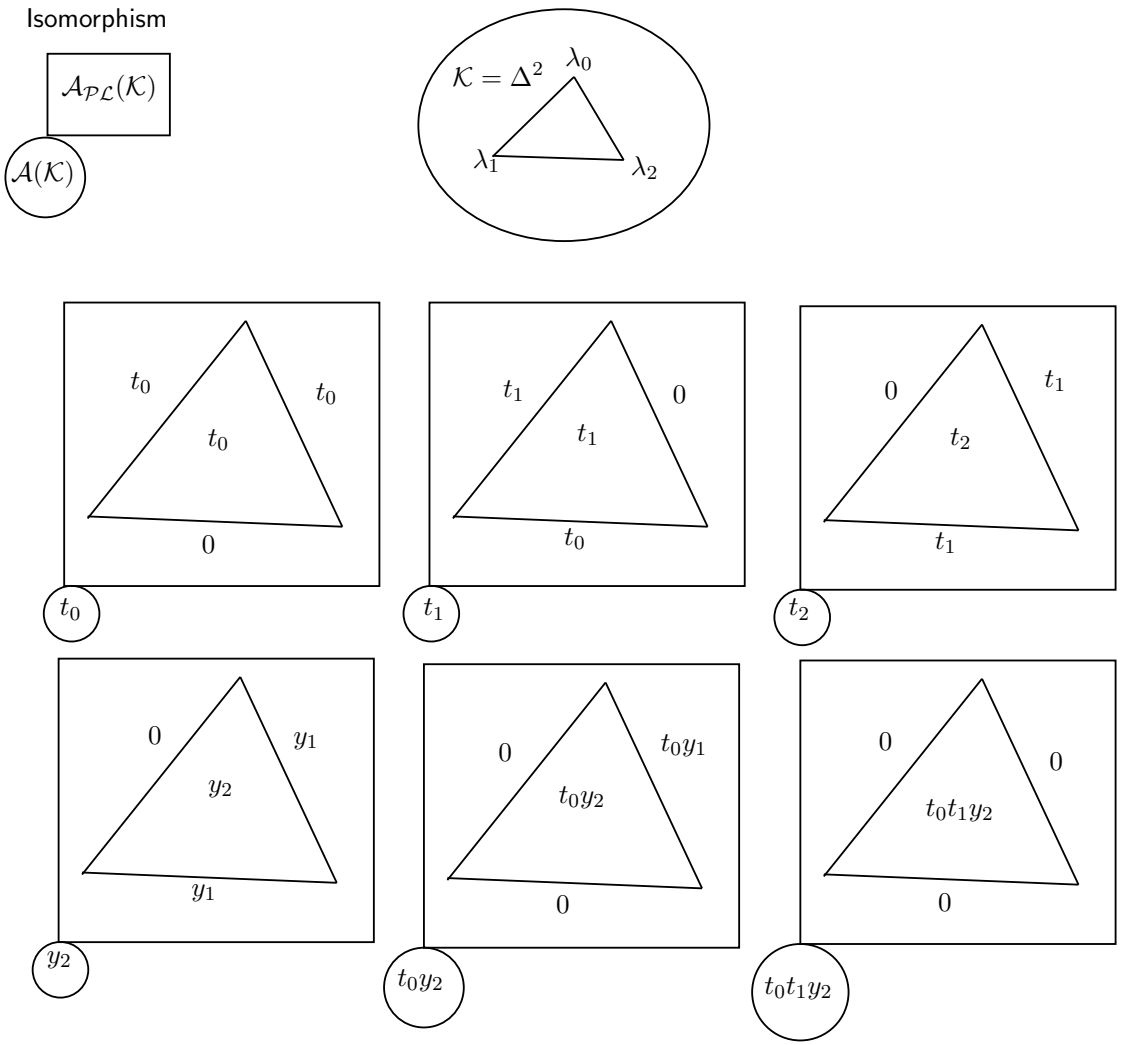
$$(A_{PL})_3 = \frac{\Lambda(t_0, \ldots, t_3, y_0, \ldots, y_3)}{\langle \sum t_i - 1, \sum y_j \rangle}$$



$$A(K) = \frac{(A_{PL})_3}{\langle I_K \rangle}$$

$$I_K = \langle t_0t_1t_2, t_0t_1y_2, t_0t_2y_1, t_1t_2y_0, t_0y_1y_2, t_1y_0y_2, t_2y_0y_1, y_0y_1y_2 \rangle$$

Method 2: the combinatorial approach



Step 8: Find a preimage by $d^i_{\mathcal{A}}$ of the exact elements

Method 3: Dupont's contraction

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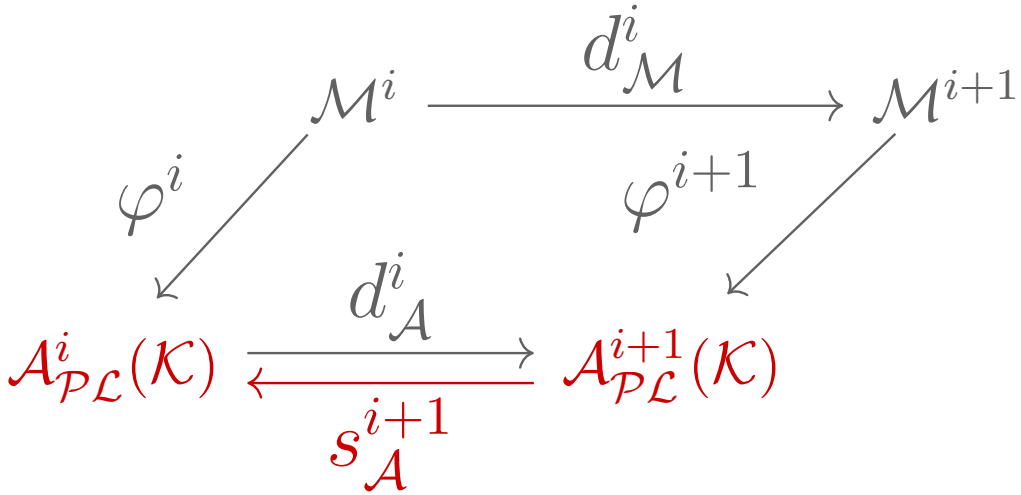
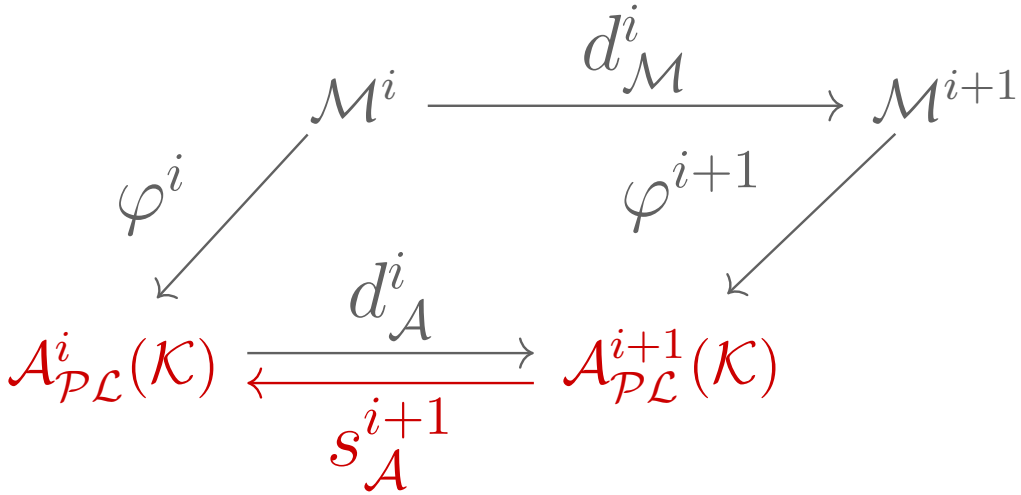


Method 3: Dupont's contraction



Suppose we have an element $\Psi \in \mathcal{A}^{i+1}_{\mathcal{PL}}(\mathcal{K})$. By Dupont's contraction we know that:

$$\Psi - (\mathcal{W} \circ \oint)(\Psi) = (s \circ d)(\Psi) + (d \circ s)(\Psi).$$

Method 3: Dupont's contraction

Additionally, suppose that $\Psi = d\theta$. Then,

$$d\theta - (\mathcal{W} \circ \oint)(d\theta) = \cancel{(s \circ d)(d\theta)} + (d \circ s)(d\theta).$$

Method 3: Dupont's contraction

Additionally, suppose that $\Psi = d\theta$. Then,

$$d\theta - (\mathcal{W} \circ \oint)(d\theta) = \cancel{(s \circ d)(d\theta)} + (d \circ s)(d\theta).$$

Moreover, we know that the differential commutes with integration and Whitney's map:

$$d(\theta) - (d \circ \mathcal{W} \circ \oint)(\theta) = (d \circ s)(d\theta).$$

Method 3: Dupont's contraction

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Then,

$$d(\theta) = d((s \circ d)(\theta) + (\mathcal{W} \circ \oint)(\theta)).$$

Method 3: Dupont's contraction

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Moreover, we know that the differential commutes with integration and Whitney's map:

$$d(\theta) - (d \circ \mathcal{W} \circ \oint)(\theta) = (d \circ s)(d\theta).$$

Then,

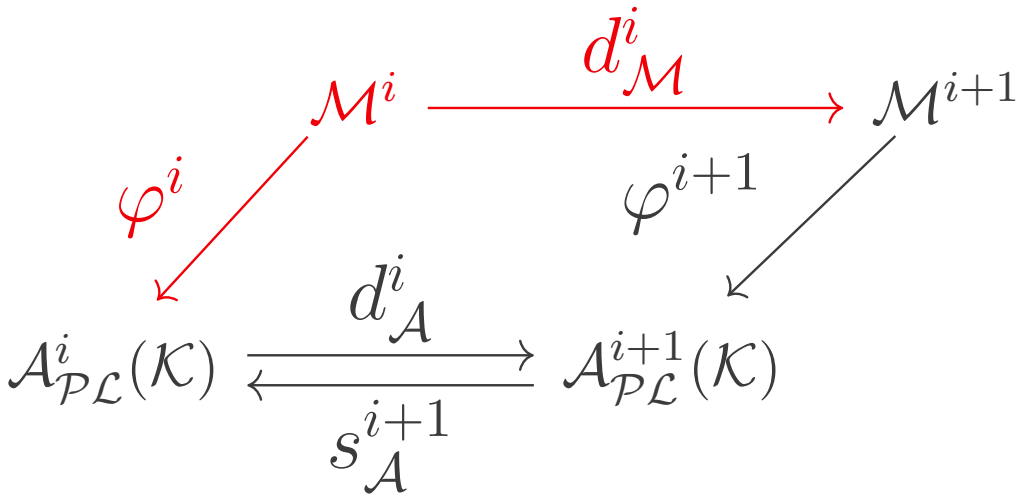
$$d(\theta) = d((s \circ d)(\theta) + (\mathcal{W} \circ \oint)(\theta)).$$

So, **the element we are looking for is:**

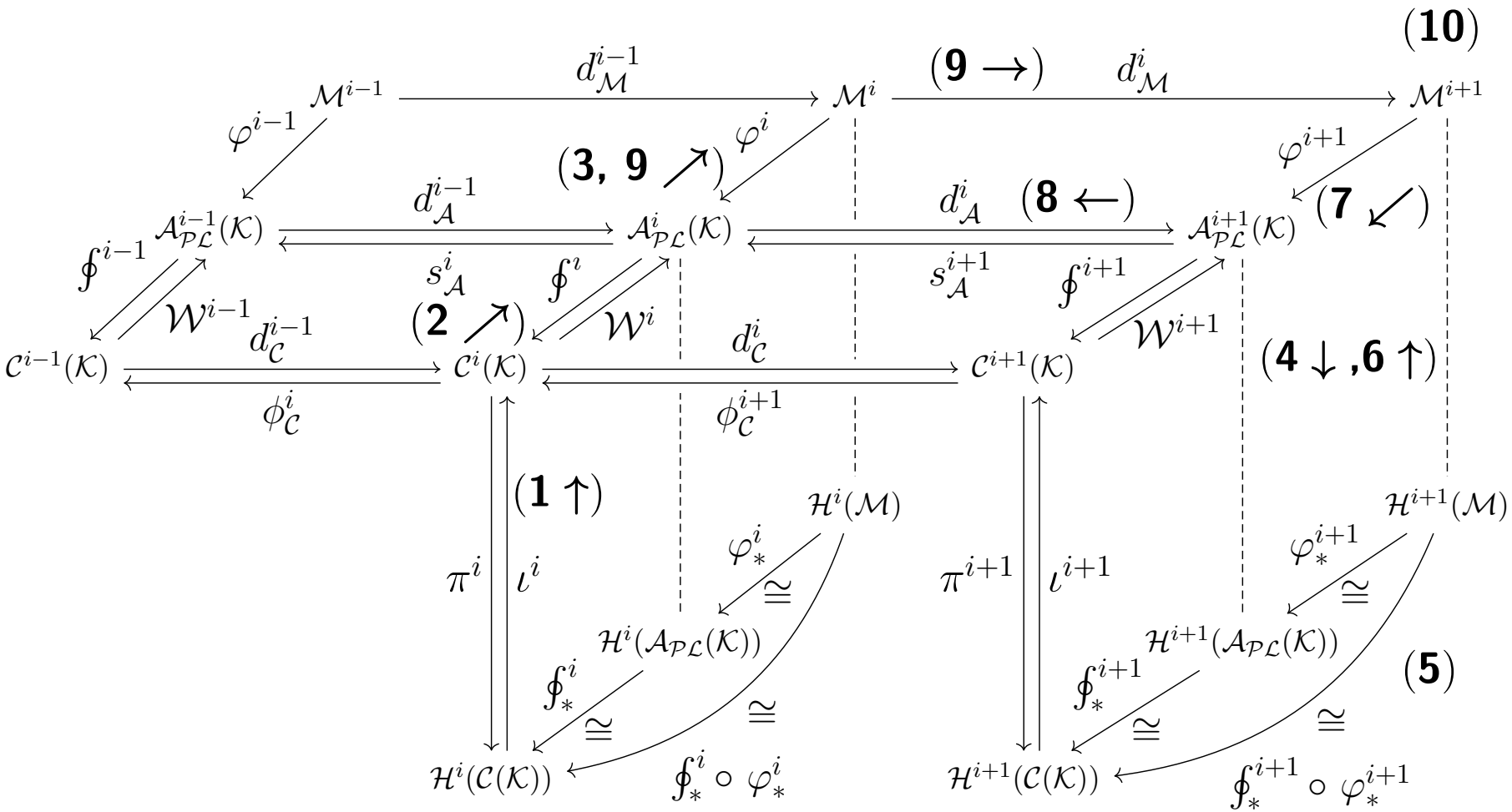
$$\theta = s(d\theta) + \mathcal{W}(\alpha),$$

where α is a solution of the linear system $d_{\mathcal{C}}(\alpha) = \oint(d\theta)$.

Step 9: Update the model $(\mathcal{M}, d_{\mathcal{M}})$



Graphical summary



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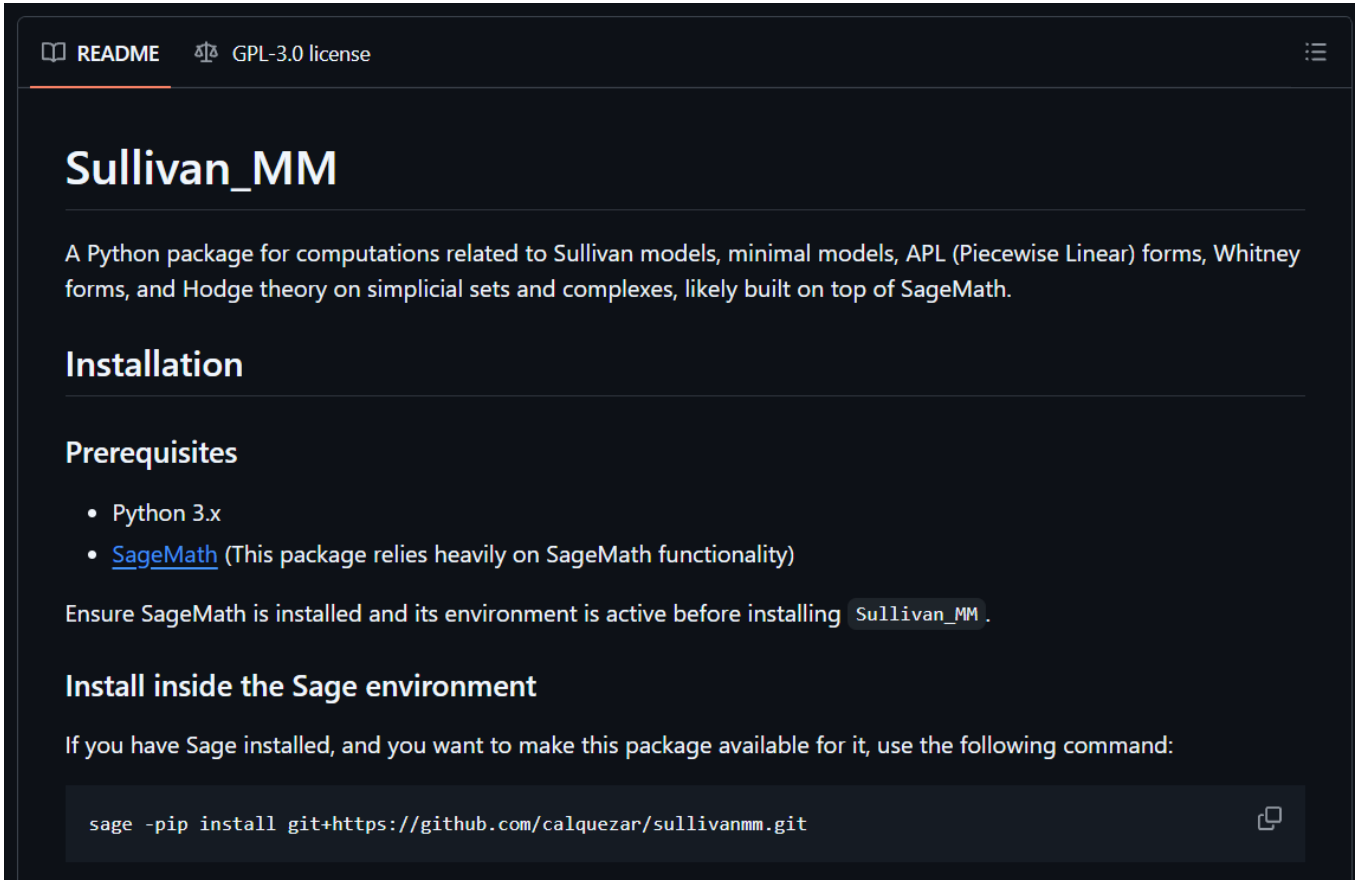
Sullivan Models package

Available at `https://github.com/calquezar`.



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Sullivan_MM

A Python package for computations related to Sullivan models, minimal models, APL (Piecewise Linear) forms, Whitney forms, and Hodge theory on simplicial sets and complexes, likely built on top of SageMath.

Installation

Prerequisites

- Python 3.x
- SageMath (This package relies heavily on SageMath functionality)

Ensure SageMath is installed and its environment is active before installing `Sullivan_MM`.

Install inside the Sage environment

If you have Sage installed, and you want to make this package available for it, use the following command:

```
sage -pip install git+https://github.com/calquezar/sullivanmm.git
```

Sullivan Models package I

```
sage: from Sullivan_MM import *
sage: S2 = simplicial_sets.Sphere(2)
sage: K = S2.wedge(S2)
sage: W = APLK(K)
```

Sullivan Models package II

```
sage: W.inject_variables()
Getting generators of degree  0 ...
Getting generators of degree  1 ...
Getting generators of degree  2 ...
Defining h0_0, h2_0, h2_1
sage: h2_0.to_AplK()
{sigma_2: 2*t2*y0*y1 - 2*t1*y0*y2 + 2*t0*y1*y2}
sage: h2_0+h2_1
[h2_0 + h2_1]
sage: _.to_AplK()
{sigma_2: 2*t2*y0*y1 - 2*t1*y0*y2 + 2*t0*y1*y2,
 sigma_2: 2*t2*y0*y1 - 2*t1*y0*y2 + 2*t0*y1*y2}
```

Sullivan Models package III

```
sage: M = W.minimal_model(i=3)
sage: M.domain()
Commutative Differential Graded Algebra with
 ↪  generators ('x2_0', 'x2_1', 'y3_0', 'y3_1',
 ↪  'y3_2') in degrees (2, 2, 3, 3, 3) over Rational
 ↪  Field with differential:
    x2_0 --> 0
    x2_1 --> 0
    y3_0 --> x2_0^2
    y3_1 --> x2_0*x2_1
    y3_2 --> x2_1^2
```

Sullivan Models package IV

```
sage: A = M.domain()
sage: A.inject_variables()
Defining x2_0, x2_1, y3_0, y3_1, y3_2
sage: M.phi(x2_0)
h2_0
sage: M.phi(x2_0) in W
True
```

Example: the complement of a Fermat curve of degree d in the complex projective space.

In a recent work [1], Artal, Larraya and Marco, give an explicit construction of a n-dimensional Δ-complex that is a strong deformation retract of the complement of a Fermat hypersurface of degree d in the complex projective space $\mathbb{CP}^n$. We tested the case of a plane Fermat curve of degree $d = 3$ in $\mathbb{CP}^2$.

Reference

[1] E. Artal, A. Larraya Sancho, M. A. Marco Buzunáriz. (2026). Retraction of the complement of smooth projective hypersurfaces to an n-dimensional Δ-complex.

Example: the complement of a Fermat curve of degree d in the complex projective space. I

```
sage: from Sullivan_MM import *
sage: from sage.topology.simplicial_set import
    ↪  SimplicialSet
sage: d = 3
sage: Kd=DeltaComplex(cells_of_Kd(d))
sage: S=SimplicialSet(Kd)
sage: WS = APLK(S)
sage: MS =
    ↪  WS.minimal_model(i=3,max_iterations=3,verbose=False,
                                    partial_result=True)
```

Example: the complement of a Fermat curve of degree d in the complex projective space. II

```
sage: MS.domain()
Commutative Differential Graded Algebra with generators ('x2_0',
'x2_1', 'y3_0', 'y3_1', 'y3_2') in degrees (2, 2, 3, 3, 3) over
Rational Field with differential:
   x2_0 --> 0
   x2_1 --> 0
   y3_0 --> x2_0^2
   y3_1 --> x2_0*x2_1
   y3_2 --> x2_1^2
```

Example: the complement of a Fermat curve of degree d in the complex projective space.

We observed that the minimal model has 2 generators of degree 2, which means that, if the space would be a Sullivan space, its second rational homotopy group would have rank 2.

Remark

Sullivan spaces are the most general setting where the Sullivan minimal model fully captures the Rational Homotopy Type of such topological spaces.

Example: the complement of a Fermat curve of degree d in the complex projective space. I

However, if we compute the minimal model of its universal cover:

```
sage: S=S.set_base_point(S.all_n_simplices(0)[0])
sage: U = S.universal_cover()
sage: WU = APLK(U)
sage: MU =
  ↪  WU.minimal_model(i=3,max_iterations=3,verbose=False,
                                  partial_result=True)
sage: A = MU.domain()
sage: len(A.basis(2))
8
```

we observe that the minimal model of the universal cover has 8 generators of degree 2.

Example: the complement of a Fermat curve of degree d in the complex projective space. I

As it is simply-connected, it must be a Sullivan space, so the π_2 of the universal cover (which coincides with the π_2 of the space itself) has rank 8. Due to the fact that:

```
1 len(MU.domain().basis(2))==len(MS.domain().basis(2))
2 [output] ==> False
```

The complement of the curve is not a Sullivan space!

As this doesn't match the rank 2 obtained before, we can conclude that the complement of the smooth cubic is not a Sullivan space.

Example: the $\mathcal{K}3$ Surface

Number of simplices of the $\mathcal{K}3$ Surface in each dimension.

	Dimension				
	0	**1**	**2**	**3**	**4**
Nº Simplices	16	120	560	720	288

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The Betti numbers of the space. Omitting degree 0, the cohomology has 22 generators of degree 2 and one generator of degree 4.

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The rank of the first homotopy groups are:

	$\pi_n(\mathcal{K}3)$			
	1	**2**	**3**	**4**
rank	0	22	252	3520

Execution times

Plain Fermat curve C of degree 3

Taking the universal cover of $\mathbb{CP}^2 \setminus \mathrm{C}$...

3-minimal model	$\sim$	5 seconds.
4-minimal model ($\lvert\pi_4(C)\rvert = 378$)	$\sim$	23 minutes.

The minimal model of the $\mathcal{K}3$ Surface

3-minimal model	$\sim$	20	minutes.
4-minimal model	$\sim$	4	hours.

Execution times

Let $\mathcal{K} = \Delta^{10}$ and consider an exact element $d\theta \in \mathcal{A}_{\mathcal{PL}}(\mathcal{K})$ of degree 6 where the polynomial differential forms on $\mathcal{K}$ have polynomial coefficients of degree 20 in $\mathbb{Q}\langle t_0, \ldots, t_n \rangle$.

Dupont's contraction

Finding θ takes around 17 minutes.

1 Introduction

2 Adaptation points of MMarco's method

3 The effective method for $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$

4 Examples

5 Conclusions

Conclusions

The main conclusions of this thesis are the following:

(C1) **The main result of this thesis** has been to develop and implement, within a computer algebra system, an effective method for the computation of the Sullivan minimal model of the algebra of polynomial differential forms $\mathcal{A}_{\mathcal{PL}}(\mathcal{K})$ of a finite type simplicial set $\mathcal{K}$.

Conclusions

(C2) **Use of a cohomological proxy:** An approach has been designed that uses $\mathcal{C}(\mathcal{K})$ and its cohomology $\mathcal{H}^*(\mathcal{C}(\mathcal{K}))$ as an indirect way to work with $\mathcal{H}^*(\mathcal{A}_{\mathcal{PL}}(\mathcal{K}))$. Through the combined use of Algebraic Topological models (AT-models) and the de Rham integration and Whitney map quasi-isomorphisms, the problem of computing $\mathcal{H}^*(\mathcal{A}_{\mathcal{PL}}(\mathcal{K}))$ is avoided, allowing generators to be lifted and injectivity/surjectivity to be verified effectively.

Conclusions

(C3) **Implementation of Dupont's contraction:** the problem of finding primitives has been solved through the concrete implementation of Dupont's simplicial contraction. Unlike alternative approaches based on matrix truncation or combinatorial models, Dupont's contraction operates locally, simplex by simplex. This method guarantees the patch conditions by definition and reduces the computational complexity respect to other approaches.

Future work

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(F4) To try to compute the rational homotopy type of non-formal Milnor fibers [Su].

Reference

[Su] Suciu, A.I. Highly connected non-formal Milnor fibers via polyhedral products. European Journal of Mathematics 12, 49 (2026).

Thank you for your attention.

